دومین سمینار تخصصی

نظریه قابلیت اعتماد و کاربردهای آن

۲۹ و ۳۰ اردیبهشت ۱۳۹۵

دانشگاه تهران

دانشکده ریاضی، آمار و علوم کامپیوتر

$$f(t)=\left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1}e^{-\left(\frac{t}{\eta}\right)^{\beta}}$$

تلفن: ۶۱۱۱۲۶۱۳-۰۲۱ فکس: ۶۶۴۱۲۱۷۸-۰۲۱ وبسایت: http://wosdce.um.ac.ir

آدرس: خیابان ۱۶ آذر- دانشگاه تهران - پردیس علوم

Second Seminar on

Reliability Theory and its Applications

18-19, May 2016

Proceedings of

2nd Seminar on

Reliability Theory and its Applications

Department of Statistics, School of Mathematics,
Statistics and Computer Science
University of Tehran, Tehran, Iran

Iranian Statistical Society

Statistical Research and Training Center

Aerospace Research Institute

and

Ordered and Spatial Data Center of Excellence
Ferdowsi University of Mashhad, Iran

18-19 May, 2016

Preface

On behalf of the organizing and scientific committees, we would like to extend a very warm welcome to all the participants of the 2nd Seminar on "**Reliability Theory and its Applications**".

Hope that this seminar provides an environment of useful discussions and would also exchange scientific ideas through opinions. We wish to express our gratitude to the numerous individuals and organizations that have contributed to the success of this seminar, in which around 100 colleagues, researchers, and postgraduate students have participated.

Finally, we would like to extend our sincere gratitude to the head of the College of Science and the head of the School of Mathematics, Statistics and Computer Science at University of Tehran for their kind cooperation. We wish them all the best.

Firoozeh Haghighi

Topics of the Seminar:

- Statistical inference based on reliability data
- Accelerated life testing
- System maintenance and repair policies
- Concepts of aging
- Stochastic ordering in reliability
- Stress- strength model
- Reliability of networks
- Survival analysis
- Quality control and reliability
- Information theory and reliability

Scientific Committee:

- Ahmadi, J. Ferdowsi University of Mashhad
- Asadi, M. University of Isfahan
- Farsi, M. A. Aerospace Research Institute
- Haghighi, F. University of Tehran (**Chairman**)
- Hamedani, A. Isfahan University of Technology
- Khaledi, B. Razi University
- Khodadadi, A. University of Shahid Beheshti
- Khorram, E. Amirkabir University of Technology
- Noorossana, R. Iran University of Science and Technology
- Parsian, A. University of Tehran

Organizing Committee:

- Amini, M. University of Tehran
- Jabbari Nooghabi, H. Ferdowsi University of Mashhad
- Pezeshk, H. University of Tehran
- Shemehsavar, S. University of Tehran

Contents

On the lifetime comparison of series systems with random number of components
Abbasi, S. , Alamatsaz, M. 1

Economic and Economic Statistical Design of $\overline{X}-$Control Charts Under a Bathtub-shaped Shock Model
Khadem, Y. , Pasha, M. A. , Moghadam, M. B. , Fani, Sh. 5

Confidence Intervals for Quantiles based on Progressive First-Failure Censored Data from Exponential Distribution
Ahmadi, Mohammad Vali 9

Stochastic orders based on quantile function in reversed time
Rezaei, M. , Akbari, M. , Jomhoori, S. 13

Stochastic comparisons of redundancy allocation at component level versus systems level
Amini-Seresht Ebrahim 17

Prediction of Lower Records Based on Hybrid Censored Samples for the Topp-Leone Distribution
Aminzadeh, M. , MirMostafaee, S.M.T.K. , Safarpour, N. 21

Three-State Networks Subject to Shocks
Ashrafi, S. , Zarezadeh, S. 25

The mean residual life and its connection to information measures
Majid Asadi 29

Bayesian Estimation for the Marshall-Olkin Extended Exponential Distribution Under Progressively Type-II Censoring with Binomial Removals
Azizi, E. , MirMostafaee, S.M.T.K. 30

Statistical Inference of Accelerated Life Tests with Masked Interval Data under Tampered Failure Rate Model
Azizi, F. , Torabi, L. 34

Inference of $R = P[X < Y]$ for proportional reversed hazard rate model
Salman Babayi , Esmaeile Khorram 38

Likelihood ratio order of maxima from heterogeneous gamma random variables
Bashkar, E. , Torabi, H. 44

A concept of extended cumulative residual entropy and its relation to reliability measures
Bazari, N. 50

Measure of uncertainty in the residual lifetime of a coherent system
Chahkandi, M., Toomaj, A. 53

Dependence structure based guidelines for active redundancy allocation problems in engineering systems
Jeddi, H. , Doostparast, M. 58

Some information measures for three-state networks
Eftekhar, S., Zarezadeh, S. 64

Reversed Preservation of NWU(3) Class for Parallel System
Gholami, E. , Salehi, E. 68

Characterization of distributions by expectation of functions of some measures in the reliability
Goodarzi, F. , Amini, M., Mohtashami Borzadaran, G.R. 72

A partially accelerated life test planning with competing risks and linear degradation path under tampered failure rate model
Haghighi, F. 76

Optimal Design of Bivariate Step Stress Model For Reliability Prediction Under Type II Censoring
Nooshin Hakamipour , Sadegh Rezaei 80

Evidences in sequential order statistics coming from a general class of distributions
Hashempour, M., Doostparast, M. 84

The Prediction of Future Record and Inter-Record Time for Two Parameters Exponential Distribution
Zakerzadeh, H. , Dastbaravarde, A., Karami, M. 90

Bayesian estimation for component distribution from system lifetime data
Kelkinnama M. 97

A Note on the reliability of a coherent system in stress-strength model
Khanjari Sadegh, M., Kasraei, F. 101

On the Fisher Information of Residual Lifetime
Kharazmi, O. , Asadi, M. 105

Stress-Strength Model in Discrete Distributions
Khorashadizadeh, M. , Mohtashami Borzadaran, G. R. 110

Estimation of the Shape Parameter of the Weighted Exponential Distribution Under the Record Ranked Set Sampling Plan
Khoshkhoo Amiri, Z. , MirMostafaee, S.M.T.K. 115

Estimating the Survival Function Using Different Copula Functions
Mireh, S. , Khodadadi, A. 119

Analysis of Masked Data with non-ignorable missing mechanism
Misaii, H. , Eftekhari, S. 123

An algorithm to transform the generalized joint signatures
Mohammadi, L. , Zarezadeh, S. 127

Optimal Burn-in Test for Bivariate Degradation Model
Mosayebi, E., Shemehsavar, S. 131

Concomitant of order statistics arising from non-identical independent random vector
Rezapour, M., Sheikhi, A., Negarestani, H. 135

Some design of strength system in stress-strength model
Pakdaman, Z., Ahmadi, J., Doostparast, M. 139

Parameters Estimation for the Generalized Inverted Exponential Distribution Based on Progressive Type II Hybrid Censored Sample
Panahi, H. 143

Optimal mixed redundancy allocation in a bridge network by a hierarchical memetic algorithm
Ramezani, Z. , Pourdarvish, A. 147

On the reliability of series and parallel systems with randomized components
Rahimi, Z. , Zarezadeh, S., Sharafi, M. 154

Stochastic Comparisons of $(n-k+1)$-out-of-n Systems with exchangeable components Based on Residual Lifetimes
Salehi, E. 158

Representing the Reliability Function in Terms of Mean Residual Lifetime of Coherent Systems
Mahdi Tavangar 162

A note on the mixture representation of the Conditional Inactivity Time of a Coherent System
Teimoori, F. , Goli, S. 166

The New Burr Distribution with Application to Bladder Cancer Susceptibility
Tondpour, Z. , Yari, GH. 170

Effect of imperfect ranking on estimator of stress-strength model based on ranked set sampling
Zamanzade, E. 174

Preventive Maintenance of a Network Subject to External Shocks
Zarezadeh, S. , Ashrafi, S. 178

Network Reliability Design Using Ranking and Selection Procedures
Mohammadi, Z. 182

Availability Optimization of Redundancy Allocation Problem for a System with Repairable and Non-Repairable Components
Zoulfaghari, H. , Ahmadi, A., Zeinal Hamadani, A. 186

Likelihood Based Inference on Progressive type-II hybrid-censored data For Burr Type III Distribution
Arabi Belaghi, R., Valizadeh Gamchi, F., Bevrani, H. 194

An Algorithm for Computing the Multidimensional signature with Multithread Parallelization
Yarahmadi, A., Zarezadeh, S. 200

Estimation of the Bivariate Burr Type III Distribution Parameters Based on Censored Samples
Zazarmi Azizi, A., Sayyareh, A. 204

Inference for Lindley distribution under different censoring schemes
Akbar Asgharzadeh 208

Mean residual life function of a coherent system with known number of failed components
Khanjari Sadegh, M., Mirjalili, A. 209

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

On the lifetime comparison of series systems with random number of components

Abbasi, S. [1] and Alamatsaz, M. [2]

[1] Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran

[2] Department of Statistics, University of Isfahan, Isfahan, Iran

Abstract

In this paper, we stochastically compare Harris family distributions having random tilt parameter with Harris family distributions having fixed tilt parameter. In reliability context, this applies to stochastic comparison of lifetime of a series system with random number of components with another similar but mixed system. Our comparison tools are various types of orderings. In addition, we obtain an upper bound for maximum error in evaluating the reliability function. We also present two bounds for a Harris family survival function conditioned on its random tilt parameter, which are useful in distinguishing failure probability of a component after a time t. Several previous findings, regarding Marshall-Olkin family follow as spacial cases of our results.

Keywords: Harris family distribution , Marshall-Olkin distribution, Shifted proportional stochastic ordering, Reliability function.

1 Introduction

To cover a wide range of data such as those with a high degree of skewness and kurtosis, Marshall and Olkin (1997) and Aly and Benkherouf (2012) introduced two families of distributions. In their approaches, they considered a baseline distribution and extended it to a new and more flexible distribution. The resulting classes are called Marshall-Olkin

[1] s_abbasi_29@yahoo.com
[2] alamatho@sci.ui.ac.ir

and Harris family of distributions, respectively. Both classes of such distributions are, in particular, useful in reliability theory. Aly and Benkherouf (2011) generated the Harris family of survival functions $\bar{H}$ as

$$\bar{H}(x;\theta,k) = \Big(\frac{\theta\bar{F}^k(x)}{1-\bar{\theta}\bar{F}^k(x)}\Big)^{1/k}, \qquad k>0, 0<\theta<\infty, \bar{\theta}=1-\theta. \tag{1}$$

The df F in Equation (Eq) (1-1) is called the baseline df and θ is called the tilt parameter. In the Harris family distribution, there is no theoretical basis for choosing the baseline distribution and the distribution of tilt parameter; when tilt parameter is a rv. Therefore, it is important to see how a Harris family rv responds to the change of the baseline distribution and tilt parameter. This paper, mainly investigates how the relations between tilt parameters effect in stochastic orders (see; Shaked and Shanthikumar (2007)) between two Harris family distributions with fix and rv tilt parameters. Considering the utility desired, we are able to select a fix or rv tilt parameter.
We also obtain an upper bound for the maximum error in evaluating the reliability function. In many practical problems, using a sample data set, we are able to obtain some life information such as the mean and variance of a life distribution. But the exact value of the reliability function can not be easily obtained. However, it is still helpful to derive some bounds for a reliability function based on the known information. These bounds can tell us the scope of the reliability of products and provide a basis for further improvements. In addition, we obtain two bounds for survival functions conditioned on the tilt random parameter, which are useful in distinguishing the failure probability of a component after a time t when the tilt parameter is unobservable.

2 Stochastic comparison under mixtures

In Eq (1.1), let the parameter Θ be an absolutely continuous rv with df $G(.)$ and pdf $g(.)$. Then, its corresponding unconditional Harris sf is given by

$$\bar{H}(x;k) \;=\; \int_0^\infty \bar{H}(x;\theta,k)g(\theta)d\theta = E[\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)}]^{\frac{1}{k}}\bar{F}(x).$$

We denote the corresponding rv by X^*. Our results enfold Nanda and Das (2012)'s findings in this connection.

Theorem 1. *Let X and X^* be two rv's with sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Then, $X \le_{lr} (\ge_{lr}) X^*$ if $P(\Theta\ge\nu)=1$ $(P(0<\Theta\le\nu)=1)$.*

In the following theorem we compare the ageing intensity ordering between a Harris family and its mixture. First, we give the following lemma. Let X be a rv with sf $\bar{H}(.;\nu,k)$ and X^* be a rv with sf $\bar{H}(.;k)$ and hazard rate $r_H(.;k)$. Then, if $P(0<\Theta\le 1)=1$, $\frac{r_H(x;k)}{r_H(x;\nu,k)}$ is decreasing in x.

Theorem 2. *Let X and X^* be two rv's with sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Then, if $P(0<\Theta\le 1)=1$ we have $X \le_{AI} X^*$.*

Theorem 3. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta\ge 1$ $(0<\Theta\le 1)$ with probability 1 and $0<\nu\le 1$ $(\nu\ge 1)$. Then*

(a) $X \leq_{plr\uparrow} (\geq_{plr\uparrow}) X^*$ *if* $X \in UIPLR$,

(b) $X \leq_{plr} (\geq_{plr}) X^*$ *if* $X \in IPLR$,

(c) $X \leq_{lr\uparrow} (\geq_{lr\uparrow}) X^*$ *if* $X \in ILR$.

Theorem 4. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ $(0 < \Theta \leq 1)$ with probability 1 and $0 < \nu \leq 1$ $(\nu \geq 1)$. Then*

(a) $X \leq_{plr\downarrow} (\geq_{plr\downarrow}) X^*$ *if* $X \in DIPLR$,

(b) $X \leq_{lr\downarrow} (\geq_{lr\downarrow}) X^*$ *if* $X \in DLR$.

Theorem 5. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ $(0 < \Theta \leq 1)$ with probability 1 and $0 < \nu \leq 1$ $(\nu \geq 1)$. Then*

(a) $X \leq_{phr\uparrow} (\geq_{phr\uparrow}) X^*$ *if* $X \in UIPHR$,

(b) $X \leq_{phr} (\geq_{phr}) X^*$ *if* $X \in IPHR$,

(c) $X \leq_{hr\uparrow} (\geq_{hr\uparrow}) X^*$ *if* $X \in IHR$.

Theorem 6. *Let X and X^* be two continuous and non-negative rv's corresponding to sf's $\bar{H}(.;\nu,k)$ and $\bar{H}(.;k)$, respectively. Also, let $\Theta \geq 1$ $(0 < \Theta \leq 1)$ with probability 1 and $0 < \nu \leq 1$ $(\nu \geq 1)$. Then*

(a) $X \leq_{phr\downarrow} (\geq_{phr\downarrow}) X^*$ *if* $X \in DIPHR$,

(a) $X \leq_{hr\downarrow} (\geq_{hr\downarrow}) X^*$ *if* $X \in DHR$.

3 Bounds for survival function

We now obtain some useful bounds concerning a tilt-mixture Harris family distribution. First, we note that for any x, $t > 0$ and $k \geq 1$, $(\frac{t}{1-(1-t)\bar{F}^k(x)})^{\frac{1}{k}}$ is a concave function of t. Thus, using Jensen's inequality, where $\mu = E(\Theta)$, we have

$$\bar{H}(x;k) = \bar{F}(x)E(\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)})^{\frac{1}{k}} \leq \bar{F}(x)(\frac{\mu}{1-(1-\mu)\bar{F}^k(x)})^{\frac{1}{k}}.$$

For a non-negative baseline rv X, we have

$$\begin{cases} E(\Theta)^{\frac{1}{k}} \leq \frac{\bar{H}(x;k)}{\bar{F}(x)} \leq 1, & P(0 < \Theta < 1) = 1 \\ 1 \leq \frac{\bar{H}(x;k)}{\bar{F}(x)} \leq E(\Theta)^{\frac{1}{k}}, & P(\Theta \geq 1) = 1 \end{cases}$$

In the following theorem we obtain an upper bound for the maximum error in evaluating the reliability function.

Theorem 7. *Let baseline rv X be non-negative and $\bar{H}_0(x;k) = e^{-\frac{x}{\mu_0}}E(\frac{\Theta}{1-\bar{\Theta}\bar{F}^k(x)})^{\frac{1}{k}}$, where $\mu_0 = E(X)$. If the baseline distribution has DMRL property, then*

$$sup_{x\geq 0} \mid \bar{H}(x;k) - \bar{H}_0(x;k) \mid \leq \begin{cases} E(\Theta)^{\frac{1}{k}}(1-e^{-1})(1-\gamma_0^2), & P(\Theta \geq 1) = 1 \\ (1-e^{-1})(1-\gamma_0^2), & P(0 < \Theta < 1) = 1 \end{cases}$$

where γ_0 is the coefficient of variation of the baseline distribution.

In the following theorem we obtain an upper bound for the failure probability of a component after a time t.

Theorem 8. *Let tilt parameter Θ have NBU(2) property and $k \geq 1$. For non-negative rv X,*

(a) *Provided that $P(0 \leq \Theta \leq 1) = 1$, we have*

$$P(X^* \geq t \mid \Theta > \vartheta) \leq min\{\bar{F}(t) + (\frac{\bar{F}^k(t)\vartheta}{1-(1-\vartheta)\bar{F}^k(t)})^{\frac{1}{k}},\ 1\}.$$

(b) *Provided that $P(\Theta \geq 1) = 1$, we have*

$$P(X^* \geq t \mid \Theta > \vartheta) \leq min\{\bar{F}(t)E(\Theta^{\frac{1}{k}}) + (\frac{\bar{F}^k(t)\vartheta}{1-(1-\vartheta)\bar{F}^k(t)})^{\frac{1}{k}},\ 1\}.$$

References

[1] Aly EAA. and Benkherouf L. (2011), A new family of distributions based on probability generating functions, *Sankhya B*, vol. 73, pp. 70-80.

[2] Marshall A. W. and Olkin I. (1997), A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika*, vol. 84, no. 3, pp. 641-652.

[3] Nanda A. k. and Das S. (2012) Stochastic orders of the MarshallOlkin extended distribution," *Statistics and Probability Letters,* vol. 82, pp. 295-302.

[4] Shaked M., Shanthikumar J. G. (2007), *Stochastic Orders.* Academic Press, New York.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Economic and Economic Statistical Design of $\overline{X}-$Control Charts Under a Bathtub-shaped Shock Model

Khadem, Y. [1] and Pasha, M. A. [2] and Moghadam, M. B. [3] and Fani, Sh. [4]

[1,2,3,4] Department of Statistics, Allameh Tabatabai University, Beheshti-Ghasir Ave., Tehran-1513615411, Iran

Abstract

The optimal design parameters of the economic and economic statistical designs of control charts depend mainly on the reliability of the system that can be regarded as the process failure mechanism, shock model of the system, lifetime distribution of the process, or the assignable cause occurence time. In this paper, we presented a cost model under a bathtub-shaped (U-shaped) failure rate lifetime distribution that starts at optimum burn-in time at the beginning of a product cycle and then remarkably decreases in the early stage toward an approximately constant hazard rate during the useful life stage. Afterwards, the failure rate starts to increase with the onset of wear out. The generalized cost model of Rahim and Banerjee (1993) is applied to achieve the optimal design parameters under both uniform and nonuniform sampling schemes.

Keywords: Economic Design, Economic Statistical Design, Process Failure Mechanism, Bathtub Failure Rate, Integrated Hazard over sampling interval.

1 Introduction

Control chart technique, originally introduced by Shewhart in 1924, is one of the basic tools in statistical process control. The purpose of control charts is to differentiate between

[1]khadem.yasin@yahoo.com
[2]mojtaba.a.pasha@gmail.com
[3]bamenimoghadam@atu.ac.ir
[4]fani.shabnam@gmail.com

the inevitable random causes and the assignable causes in the process. Correspondingly, the necessary corrective action may be taken before a large number of nonconforming products are manufactured.

The control limits of an $\overline{X}-$control chart which is the most popular charts in the literature to control normal process means are set at $\pm L$ standard deviations of the target mean, i.e. $\mu \pm L\frac{\sigma}{\sqrt{n}}$ where the standard deviation assumes fix along time. A sample of size n is taken from the output of the process at time intervals of h_j hours for $j = 1, 2, ...$ and then $\overline{X}$ is plotted on the chart. Observing of sample means outside the control limits of the chart is a signal that is regarded as an indication that the process is out of control. Design of control charts is the determination of the parameters n, L, and h_j that can be obtained economically and statistically optimal applying economic statistical design procedure.

The key assumption for an economic design or economic statistical design is that the in-control time of the process obeys a lifetime distribution as the process failure mechanism. Duncan (1956) and Lorenzen and Vance (1986) assumed that the occurrence time of an assignable cause follows an exponential distribution which has identified by memoryless property and constant hazard rate. This assumption has been adopted widely by many subsequent researchers. For systems with increasing failure rate, Banerjee and Rahim (1988) extended Duncan's model to the Weibull shock model using non-uniform sampling scheme in which the frequency of sampling increases with the age of the system. Rahim and Banerjee (1993) developed their previous work to a general shock model that is flexible enough to be applied for any sampling scheme.

It is now widely believed that many products, particularly electronic items such as silicon integrated circuits, exhibit a bathtub (U-shaped) failure rate function. This belief that the bathtub model plays an important role in reliability is supported by much experience and extensive data collection in many industries. Consider a system in which the operators are beginners, so the hazard rate decreases with increasing skills, then while almost constant and finally increases in depreciation components.

2 The Model Construction

A generalized economic model for any lifetime distribution function F and any sampling scheme is presented in Rahim and Banerjee (1993). Random samples of size n are drawing at time ω_j for $j = 1, 2, ..., m$; provided that $\omega_j = \sum_{i=1}^{j} h_i$ and $\lim_{m\to\infty} F(\omega_m) = 1$. In addition, p_j is defined as the conditional probability that the process shifts to the out-of-control state during the time interval (ω_{j-1}, ω_j) given that the process was at the in-control state at time ω_{j-1}. i.e. $p_j = \int_{\omega_{j-1}}^{\omega_j} f(t)dt / \int_{\omega_{j-1}}^{\infty} f(t)dt$. Define $p_0 = 0$. Therefore, the jth sampling interval will be corresponded by a pair (h_j, p_j) for $j = 1, 2, ..., m$. They proved the following expressions are satisfied:

$$E(T) = \sum_{j=1}^{\infty} h_j \prod_{i=1}^{j-1} (1-p_i) + \alpha Z_0 \sum_{j=1}^{\infty} \prod_{i=1}^{j} (1-p_i)$$
$$+\beta \sum_{j=1}^{\infty} p_j \prod_{i=0}^{j-1} (1-p_i) \sum_{i=j+1}^{\infty} h_i \beta^{i-j-1} + Z_1$$

and

$$E(C) = D_0 \sum_{j=1}^{\infty} h_j \prod_{i=1}^{j-1}(1-p_i) + \alpha Y \sum_{j=1}^{\infty} \prod_{i=1}^{j}(1-p_i) + (D_0 - D_1)\tau$$
$$+ (D_1 - D_0) \sum_{j=1}^{\infty} \omega_j p_j \prod_{i=0}^{j-1}(1-p_i) + D_1 \beta \sum_{j=1}^{\infty} p_j \prod_{i=0}^{j-1}(1-p_i) \sum_{i=j+1}^{\infty} h_i \beta^{i-j-1}$$
$$+ (a+bn)[\frac{\beta}{1-\beta} + \sum_{i=0}^{\infty} \prod_{j=1}^{i}(1-p_j)] + W$$

where Z_0 = Expected assignable cause search time for a false alarm, Z_1 = Expected time to identify the assignable cause and repair the process, a = Fixed cost per sample, b = Variable cost per sample, D_0 = Hourly cost due to nonconformities produced while in control, D_1 = Hourly cost due to nonconformities produced while out of control, Y = Cost per false alarm, W = Cost for locating and repairing the assignable cause, and τ = the expected in-control duration.

Furthermore, since normality of output characteristic, $\alpha = \Pr(\text{exceeding control limits}|\text{process in control}) = 2\Phi(-L)$, $\beta = \Pr(\text{not exceeding control limits}|\text{process out of control}) = \Phi(L - \delta\sqrt{n}) - \Phi(-L - \delta\sqrt{n})$ which Φ is the standard normal distribution function.

Now assume that

$$R(t) = 1 - F(t) = \frac{\exp(-kt^2/2)}{(1+\nu t)^{\theta/\nu}}; \qquad t \geq 0, k > 0, \nu > 0, \theta > 0$$

For $0 < k < \theta\nu$, the above lifetime distribution has a bathtub-shaped failure rate. This corresponds to the three distinct phases of a system, that is, early life, useful life, and wear-out. Combinations of different parameters of this distribution is applied for numerical investigation of the cost model.

3 Numerical Illustrations

In this section, the following numerical study is conducted to illustrate the performance of the model for determining the optimal design parameters. Assume $Z_0 = 0.25$, $Z_1 = 1$, $a = 4$, $b = 1.2$, $D_0 = 50$, $D_1 = 950$, $Y = 500$, $W = 1100$. The results are presented in table 1 and table 2 to compare economic and economic statistical design parameters under uniform and nonuniform sampling schemes.

Table 1. Economic Design Parameters

Distribution Parameters			Nonuniform Sampling Scheme						Uniform Sampling Scheme					
θ	ν	k	n	h_1	L	α	$1-\beta$	ECT	n	h	L	α	$1-\beta$	ECT
0.5	0.1	0.01	36	0.591	1.999	0.046	0.841	509.6	35	0.657	2.002	0.045	0.830	509.1
0.9	0.1	0.01	32	0.521	1.875	0.061	0.829	651.1	32	0.561	1.879	0.060	0.829	650.4
0.1	0.1	0.001	41	0.961	2.164	0.030	0.850	206.6	41	1.384	2.167	0.030	0.849	206.4
0.4	0.1	0.001	38	0.585	2.047	0.041	0.849	435.8	37	0.733	2.052	0.040	0.839	435.0

Table 2. Economic Statistical Design Parameters

Distribution Parameters			Nonuniform Sampling Scheme						Uniform Sampling Scheme					
θ	ν	k	n	h_1	L	α	$1-\beta$	ECT	n	h	L	α	$1-\beta$	ECT
0.5	0.1	0.01	42	0.668	1.960	0.050	0.900	510.7	42	0.747	1.960	0.050	0.900	510.4
0.9	0.1	0.01	42	0.588	1.960	0.050	0.900	653.0	42	0.637	1.960	0.050	0.900	652.6
0.1	0.1	0.001	47	1.065	2.133	0.033	0.902	207.1	47	1.543	2.134	0.031	0.902	207.0
0.4	0.1	0.001	43	0.655	2.006	0.045	0.900	436.7	43	0.830	2.005	0.045	0.900	436.2

References

[1] Banerjee, P. K. and Rahim, M. A. (1988) Economic Design of $\overline{X}-$Control Charts Under Weibull Shock Models, *Technometrics*, **30**, 407-414.

[2] Duncan, J. (1956) The Economic Design of $\overline{X}-$Charts Used to Maintain Current Control of a Process, *J. American Statistical Association*, **50**, 228-242.

[3] Hjorth, U. (1980) A Reliability Distribution With Increasing, Decreasing, Constant and Bathtub-shaped Failure rates, *Technometrics*, **22**, No. 1, 99-107.

[4] Lorenzen, T. J. and Vance, L. C. (1986) The Economic Design of Control Charts: A Unified Approach, *Technometrics*, **28**, 3-10.

[5] Rahim, M. A. and Banerjee, P. K. (1993) A Generalized Model for Economic Design of $\overline{X}-$Control Charts for Production Systems with Increasing Failure Rate and Early Replacement, *Naval Research Logistics*, **40**, 787-809.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Confidence Intervals for Quantiles based on Progressive First-Failure Censored Data from Exponential Distribution

Ahmadi, Mohammad Vali [1]

[1] Department of Statistics, University of Bojnord,, P. O. Box 1339, Bojnord 94531, Iran

Abstract

Confidence intervals of the quantiles provide useful information on the plausible range of them. In this paper, the shortest possible confidence interval based on the progressive first-failure censored data coming from two-parameter exponential distribution within a class of two-sided confidence intervals is constructed. This shortest confidence interval is always shorter than the corresponding equal-tail confidence interval.

Keywords: Confidence Interval , Quantiles, Progressive First-Failure Censoring Scheme, Exponential Distribution, Pivotal Quantity

1 Introduction

Censoring schemes are used to reduce the costs of experiments and to accelerate the performing of the design. There are various types of censoring schemes in the analysis of lifetime experiments where one of them is the first-failure censoring scheme, introduced in [4] and further illustrated in [2]. A generalization of first-failure censoring is progressive first-failure censoring proposed by Wu and Kus [9], which allows for units to be removed from the test at points other than the final termination point. The description of the progressive first-failure censoring is as follows. Suppose that n disjoint groups with k units within each group are put on a test at time zero. As soon as the occurrence of the i failure, R_i (for $i = 1, \ldots, m-1$) groups randomly selected and the group in which the first failure is observed are removed from the test. When the m-th failure occurs, all of the remaining groups are removed from the test.

[1] mv.ahmadi@ub.ac.ir

The two-parameter exponential distribution is one of the applied distributions for modeling lifetime data. For an excellent survey on the exponential distribution we refer the reader to the book by Balakrishnan and Basu [1]. Suppose that unit lifetimes follow the two-parameter exponential distribution, denoted by $\text{Exp}(\mu, \sigma)$, with PDF and CDF, respectively

$$f_\theta(x) = \frac{1}{\sigma} \exp\left(-\left\{\frac{x-\mu}{\sigma}\right\}\right), \quad x > \mu, \quad \mu \in \mathbb{R}, \ \sigma > 0, \tag{1}$$

and

$$F_\theta(x) = 1 - \exp\left(-\left\{\frac{x-\mu}{\sigma}\right\}\right), \quad x > \mu, \quad \mu \in \mathbb{R}, \ \sigma > 0, \tag{2}$$

where $\theta = (\mu, \sigma)$. For a given $p \in (0,1)$, the p-th quantile of this distribution $Q := Q(\mu, \sigma, p)$ is defined by the equation $\int_{-\infty}^{Q} f_\theta(x)\,\mathrm{d}x = p$ and is given by $Q = \mu - \sigma \ln(1-p)$.

This paper considers the construction of confidence intervals for Q based on the progressive first-failure censored data. The problem of inferences on the two-parameter exponential model has been considered by several researchers. See, for example, Roy and Mathew [7], Fernandez [3], Krishnamoorthy and Mathew [6] and Hayter [4].

The rest of this paper has been organized as follows. Section 2 contains some pertinent distributional results and Section 3 illustrates the construction of one-sided and two-sided confidence intervals for Q, including the equal-tail and shortest confidence intervals.

2 Some Distributional Results

Let $X_1 < X_2 < \cdots < X_m$ be the progressive first-failure censored sample from a continuous population with PDF and CDF $f_\theta(\cdot)$ and $F_\theta(\cdot)$, respectively. Following [9], the associated likelihood function of the observed data $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ reads

$$L(\theta; \mathbf{x}) = C \ k^m \prod_{i=1}^{m} f_\theta(x_i)[1 - F_\theta(x_i)]^{k(R_i+1)-1}, \quad x_1 < \cdots < x_m < \infty \tag{3}$$

where $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - \sum_{i=1}^{m-1} R_i - m + 1)$.

Upon substituting (1) and (2) into (3), the likelihood function becomes as

$$L(\theta; \mathbf{x}) = C \ \left(\frac{k}{\sigma}\right)^m \exp\left(-\frac{k}{\sigma} \sum_{i=1}^{m} (R_i + 1)(x_i - \mu)\right), \tag{4}$$

where $\mu < x_1 < \cdots < x_m$ and $\theta = (\mu, \sigma)$. The maximum likelihood estimates of μ and σ is readily derived from (4) as $\hat{\mu} = X_1$ and $\hat{\sigma} = k \sum_{i=1}^{m} (R_i + 1)\ (X_i - X_1)/m$.

Let

$$\begin{aligned}
U_1 &= nW_1, \\
U_2 &= (n - R_1 - 1)(W_2 - W_1), \\
&\vdots \\
U_m &= (n - R_1 - \cdots - R_{m-1} - m + 1)(W_m - W_{m-1}),
\end{aligned} \tag{5}$$

where $W_i = k(X_i - \mu)/\sigma$ for $i = 1, \ldots, m$.

Thomas and Wilson [8] shown that the spacings $U_1, \ldots, U_m$, as defined in (5), are independent and identically distributed as $\text{Exp}(0, 1)$. Hence, $2N(\hat{\mu} - \mu)/\sigma = 2U_1 \sim \chi^2_2$

and $2m\hat{\sigma}/\sigma = 2\sum_{i=2}^{m} U_i \sim \chi^2_{2m-2}$ where χ^2_a denotes the chi-square distribution with a degrees of freedom and $N = n \times k$. Moreover, it is clear $\hat{\mu}$ and $\hat{\sigma}$ are independent. From these results, it follows that $Z := (\hat{\mu} - Q)/(2m\hat{\sigma})$ is a pivotal quantity and its CDF is given by

$$G_\theta(z) = \begin{cases} 1 - \frac{r^N}{(2Nz+1)^{m-1}} & , \quad z \geq 0, \\ P(m-1, \frac{s}{z}) - \frac{r^N}{(2Nz+1)^{m-1}} P(m-1, \frac{s}{z} + N\ln(r)) & , \quad z < 0. \end{cases}$$

where $r = 1 - p$, $s = \ln(r)/2$ and $P(a, x) = \int_0^x t^{a-1}\mathrm{e}^{-t}\,\mathrm{d}x/\Gamma(a)$ with $\Gamma(\cdot)$ being the complete gamma function.

3 Confidence Intervals for Q

In this section, a class of confidence intervals for Q is generated, and then special cases within this class are considered.

If z_α be the α-th quantile of Z, which satisfies $G_\theta(z_\alpha) = \alpha$, from the definition of Z, it follows that

$$\begin{aligned} 1 - \alpha_1 - \alpha_2 &= \mathrm{P}(z_{\alpha_1} \leq Z \leq z_{1-\alpha_2}) \\ &= \mathrm{P}(\hat{\mu} - 2m\hat{\sigma} z_{1-\alpha_2} \leq Q \leq \hat{\mu} - 2m\hat{\sigma} z_{\alpha_1}) \end{aligned}$$

for any $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ satisfying $\alpha_1 + \alpha_2 < 1$. Consequently, a $100(1 - \alpha_1 - \alpha_2)\%$ confidence interval for Q is

$$[\hat{\mu} - 2m\hat{\sigma} z_{1-\alpha_2}, \hat{\mu} - 2m\hat{\sigma} z_{\alpha_1}]. \tag{6}$$

Different choices of α_1 and α_2 generate a class of confidence intervals for Q of this form.

3.1 One-sided Confidence Intervals for Q

Putting $\alpha_1 = 0$ (and hence $z_{\alpha_1} = -\infty$) and $\alpha_2 = \alpha$ in (6), a $100(1-\alpha)\%$ lower confidence bound for Q as $[\hat{\mu} - 2m\hat{\sigma} z_\alpha, \infty)$ is obtained. Similarly, Putting $\alpha_1 = \alpha$ and $\alpha_2 = 0$ (and hence $z_{1-\alpha_2} = \infty$) in (6), a $100(1-\alpha)\%$ upper confidence bound for Q as $(-\infty, \hat{\mu} - 2m\hat{\sigma} z_\alpha]$ is obtained.

3.2 An Equal-tail Two-sided Confidence Interval for Q

Putting $\alpha_1 = \alpha_2 = \alpha/2$ in (6), a $100(1-\alpha)\%$ equal-tail two-sided confindence interval for Q is obtained as $[\hat{\mu} - 2m\hat{\sigma} z_{1-\alpha/2}, \hat{\mu} - 2m\hat{\sigma} z_{\alpha/2}]$.

3.3 The Shortest Two-sided Confidence Interval for Q Within the class

It is clear that $[\hat{\mu} - 2m\hat{\sigma} z_{1-\alpha^*}, \hat{\mu} - 2m\hat{\sigma} z_{\alpha-\alpha^*}]$ is a $100(1-\alpha)\%$ confidence interval for Q for any $\alpha^* \in (0, \alpha)$. The length of this confidence interval is $2m\hat{\sigma}(z_{1-\alpha^*} - z_{\alpha-\alpha^*})$. The value α^* is selected such that minimizes this length. For this purpose the PDF of Z will be equal at $z_{1-\alpha^*}$ and $z_{\alpha-\alpha^*}$. Moreover, the skewness of the distribution of Z enables this confidence interval to be shorter than the equal-tail confidence interval.

References

[1] Balakrishnan, N. and Basu, A. P. (1995) *The Exponential Distribution: Theory, Methods and Applications*, Gordon and Breach, Amsterdam.

[2] Balasooriya, U. (1995) Failure-censored reliability sampling plans for the exponential distribution, *Journal of Statistical Computation and Simulation*, **52**, 337-349.

[3] Fernandez, A. J. (2007) On calculating generalized confidence intervals for the two-parameter exponential reliability function, *Statistics*, **41**, 129-135.

[4] Hayter, A. J. (2012) Confidence bands for the reliability function of a two-parameter exponential model, *Journal of Quality Technology*, **44**, 155-160.

[5] Johnson, L. G. (1964) *Theory and Technique of Variation Research*, Elsevier, Amsterdam.

[6] Krishnamoorthy, K. and Mathew, T. (2009) *Statistical Tolerance Region: Theory, Applications, and Computation*, John Wiley, New York.

[7] Roy, A. and Mathew, T. (2005) A generalized confidence limit for the reliability function of a two-parameter exponential distribution, *Journal of Statistical Planning and Inference*, **128**, 509-517.

[8] Thomas, D. R. and Wilson, W. M., (1972) Linear order statistic estimation for the two-parameter Weibull and extreme value distributions from type-II progressively censored samples, *Technometrics*, **14**, 679-691.

[9] Wu, S. J. and Kus, C. (2009) On estimation based on progressive first-failure censored sampling, *Computational Statistics and Data Analysis*, **53**, 3659-3670.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Stochastic orders based on quantile function in reversed time

Rezaei, M. [1] and Akbari, M. [2] and Jomhoori, S. [3]

[1,2,3] Department of Statistics, School of Mathematics and Statistics, University of Birjand, P.O.Box: 97175-615, Birjand, Iran

Abstract

In this note, two new stochastic orders are defined based on quantile function in reversed time. We define reverse hazard quantile order and reverse mean residual quantile order and derive some relationship between this orders and some other orders of distribution. Also, a characterization of decreasing reverse hazard quantile class is presented.

Keywords: DRHR class, Quantile function, Reversed hazard quantile order, Reversed mean residual quantile order.

1 Introduction

Quantile-based reliability analysis is to make use of quantile functions as models in lifetime data analysis. In the literature, most of the aging concept were defined in terms of measures based on distribution function. Quantile-based definitions is essential when we use quantile model for analysis of lifetime data. Generalized-Lambda distribution [9], Lambda-Tukey distribution [2], Power-Pareto distribution [3, 4], a new model proposed by Van Staden and Loots [11] and the Govindarajulu distribution [8] are identified distribution that is used for quantile-based lifetime analysis. A special properties of these families is that their distribution functions are not available in closed forms to enable the distribution-based analysis. Importance of different Lambda families for modeling failure

[1]mjrezaei@birjand.ac.ir
[2]mahboobe_akbari@birjand.ac.ir
[3]sjomhoori@birjand.ac.ir

data are demonstrated by Karian and Dudewic [5].
Let X be a nonnegative random variable with distribution function $F(x)$, satisfying $F(0) = 0$, $F(x)$ is continuous and strictly increasing and quantile function $Q(u) = \inf\{x,\ F(x) \geq u\}$. The random variable $X_{(t)} = (t - X|X \leq t)$ is called the inactivity time or reversed residual life of X. In reliability theory, stochastic orders are employed to compare of two distributions in terms of their characteristics. Vineshkumar et al [12] studied stochastic order using quantile-based reliability measures are introduced by Nair and Sankaran [6] . In this context, we define two new ordering of random variable $X_{(t)}$ in terms of quantile function and study some connection of this stochastic order with some others well-known stochastic order. Decreasing reversed hazard quantile function (DRHQ) class of distribution has been studied by Nair et al. [7]. we get a characterization of this class.

We recall definition of reversed hazard quantile function of X and reversed mean residual quantile function as follow, respectively,

$$A_X(u) = (uq_X(u))^{-1},$$

and

$$R_X(u) = u^{-1}\int_0^u t\ q_X(t)dt = u^{-1}\int_0^u (Q_X(u) - Q_X(t))dt,$$

where $Q_X(u)$ is quantile function of X and $q_X(u) = Q'_X(u)$ is quantile density function. Also, according to [10], we have the following partial orders to be used throughout the paper.

- X is smaller than Y in the right spread (excess wealth order) order ($X \leq_{RS} Y$) if $\int_u^1 (1-p)q_X(p)\ dp \leq \int_u^1 (1-p)q_Y(p)dp$ for $u \in (0,1)$ or equivalently $\int_u^1 (Q_X(p) - Q_X(u))\ dp \leq \int_u^1 (Q_Y(p) - Q_Y(u)$.
- X is smaller than Y in the Laplace transform order ($X \leq_{Lt} Y$) if $\int_0^1 (1-p)e^{-sQ_X(p)} \leq \int_0^1 (1-p)e^{-sQ_Y(p)}$

2 Main results

Definition 1. *Let X and Y be two nonnegative random variable. We say that X is smaller than Y in reversed hazard quantile function order(denoted by $X \leq_{RHQ} Y$) if*

$$A_X(u) \leq A_Y(u), \quad for\ all\ \ u \in (0,1)$$

Note that, in general, neither the stochastic orders reversed hazard rate ($\leq_{RHR}$) (cf. [10]) and $\leq_{RHQ}$ are equivalent nor one implied the other. From corollary 4 in [1], a necessary and sufficient condition to equivalence two orders $\leq_{RHR}$ and $\leq_{RHQ}$ is that F or G is $IRHR$ and $DRHR$.

Theorem 1. *If $X \leq_{RHQ} Y$ then $X \geq_{st} Y$.*

Proof. Let $X \leq_{RHQ} Y$ then $q_X(u) \geq q_Y(u)$. By integration, we have $Q_X(u) \geq Q_Y(u)$ that follows the result. □

In general case, the converse is not true. Under the condition, $\frac{Q_Y(u)}{Q_X(u)}$ is decreasing in u we have $X \leq_{RHQ} Y \Leftrightarrow X \geq_{st} Y$. It is easy to show that if $X \leq_{RHQ} Y$ then $X \geq_{RS} Y$. Also if $X \leq_{RHQ} Y$, according to 1 we have $X \geq_{st} Y$ i.e. $Q_x(t) \geq Q_Y(t)$ that implied $X \leq_{Lt} Y$.

Similarly we can define reversed mean residual quantile order as the following.

Definition 2. *Let X and Y be two nonnegative random variable. We say that X is smaller than Y in reversed mean residual quantile order(denoted by $X \leq_{RMQ} Y$) if*

$$R_X(u) \geq R_Y(u), \quad for\ all \quad u \in (0,1)$$

Straightforwardly, the reversed hazard quantile order implied the reversed mean quantile order, but the converse is not necessarily true. The next result, gives a condition under which $X \leq_{RMQ} Y \Leftrightarrow X \leq_{RHQ} Y$.

Theorem 2. *Let X and Y be two nonnegative continuous random variable with differentiable RMQ function R_X and R_Y, respectively. Suppose that $R_Y(u) - R_X(u)$ be increasing in u. Then*

$$X \leq_{RMQ} Y \Leftrightarrow X \leq_{RHQ} Y.$$

Proof. Let X and Y have reversed hazard quantile function $A_X(u) = (uq_X(u))^{-1}$ and $A_Y(u) = (uq_Y(u))^{-1}$, respectively. From the relationship between reversed hazard quantile function and reversed mean residual quantile function, we have

$$(A_X(u))^{-1} = R_X(u) + uR'_X(u),$$

where R'_X denotes the derivative of R. Similarly,

$$(A_Y(u))^{-1} = R_Y(u) + uR'_Y(u).$$

Using the increasing property of $R_Y(u) - R_X(u)$ implies that

$$(A_X(u))^{-1} = R_X(u) + uR'_X(u) \leq R_Y(u) + uR'_Y(u) = (A_Y(u))^{-1}$$

that is, $X \leq_{RHQ} Y$. □

The next result characterize DRHQ class of distribution.

Theorem 3. *If $X_{(t_1)} \leq X_{(t_2)}$ for $t_1 \leq t_2$ then distribution is belong to DRHQ class.*

Proof. Quantile function of random variable $X_{(t)}$ is $Q(u) - Q(u_0(1-u))$ where $Q(.)$ is quantile function X and $Q(u_0) = t$. By the assumption $X_{(t_1)} \leq X_{(t_2)}$ we have

$$\frac{Q(u_1) - Q(u_1(1-u))}{u_1 u_2 u} \leq \frac{Q(u_2) - Q(u_2(1-u))}{u_1 u_2 u}$$

for $u_1 = Q_X^{-1}(t_1) \leq u_2 = Q_X^{-1}(t_2)$. Then by limiting

$$u_1 q(u_1) \leq u_2 q(u_2).$$

Therefore the theorem is proved. □

References

[1] Bartoszewicz, J.(1997), Dispersive functions and stochastic orders, *Applicationes Mathematicae*, **24**, 429–444.

[2] Freimer, M., Mudholkar, G. S., Kollia, G., and Lin, C. T. (1988), A study of the generalized Tukey lambda family, *Communications in Statistics-Theory and Methods*, **17**, 3547–3567.

[3] Gilchrist, W.G (2000), *Statistical Modelling With Quantile Functions*, Chapman and Hall/CRC Press, Boca Raton.

[4] Hankin, R. K. S. and Lee, A. (2006), A new family of non-negative distributions, *Australia and New Zealand Journal of Statistics*, **48**, 67–78.

[5] Karian, Zaven A. and Dudewicz, Edward J. (2000), *Fitting statistical distributions: the Generalized Lambda Distribution and Generalized Bootstrap methods*, Chapman & Hall.

[6] Nair, N. U. and Sankaran, P. G. (2009), Quantile-Based Reliability Analysis, *Communications in Statistics-Theory and Methods*, **38**(2), 222–232.

[7] Nair, N.U., Sankaran, P.G. and Balakrishanan, N. (2013), *Quantile Based Reliability Analysis*, Birkha üser, New York.

[8] Nair, N. U., Sankaran, P. G.,and Vineshkumar, B. (2012), Govindarajulu distribution: some properties and applications, *Communications in StatisticsTheory and Methods*, **41**, 43914406.

[9] Ramberg, J. S. and Schmeiser, B. W. (1974), An approximate method for generating asymmetric random variables, *Comm. Assoc. Comput. Mach.*, **17**, 78–82.

[10] Shaked, M., Shanthikumar, J.G. (2007), *Stochastic Orders*, Springer, New York.

[11] Van Staden, P.J. and Loots, M.T. (2009), L-moment estimation for the generalized lambda distribution, In: Third Annual ASEARC Conference, New Castle, Australia.

[12] Vineshkumar, B., Unnikrishnan Nair, N. and Sankaran, P.G.,(2015), Stochastic orders using quantile-based reliability functions, *journal of the Korean Statistical Society*, **44**, 221231.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Stochastic comparisons of redundancy allocation at component level versus systems level

Amini-Seresht Ebrahim [1]

[1] Department of Statistics, Bu-Ali Sina University, Hamedan, Iran

Abstract

In this paper, we consider a coherent system of n components and n active spares. Then, we investigate stochastic comparisons of the lifetimes of series-parallel systems and discuss that component redundancy offer greater reliability than the system redundancy with respect to the hazard rate order and the reversed hazard rate order for two scenarios, matching spares and non-matching spares.

Keywords: Stochastic orders; Coherent systems; System redundancy; Active redundancy.

1 Introduction

In some applications, one way of improving the reliability of a system is the allocation of an active redundancy. On the other words, the allocation of an active redundancy is the efficient method to add redundancy components to a system. To allocate the spare components (component) to the system reliability, one natural question that arises in this direction is that, is it better to allocate the spare components (component) in parallel or series with the weakest components (component) of the system?. In general, there are two commonly used types of redundancies called active redundancy and standby redundancy commonly used in reliability engineering and system security. For active redundancy, available spares are put in parallel to components of the system and these spares start functioning simultaneously as original components. For standby redundancy, spares are attached to components of the system in a way that a spare starts functioning right after the component to which it is attached failed. In two cases, for the measurement of the performance of different allocations to the system reliability, we use the various types of the stochastic orders. For some recent results on stochastic comparisons in system

[1] e.amini64@yahoo.com

reliabilities with active redundancy and standby redundancy, one can refer to Boland et al. (1992), Singh and Misra (1994), da Costa Bueno and do Carmo (2007), Zhao, Chan, and Ng (2012) and Laniado and Lillo (2014) and the references therein.

Boland and El-Neweihi (1995) concluded that, in the sense of the usual stochastic order, redundancy at component level is better than redundancy at system level for series systems, while the reverse is true for parallel systems. Consider a system with two components and two active spares with the random lifetimes X_1, X_2 and Y_1, Y_2, respectively. Consider the lifetimes of two coherent systems as $U_1 = \min\{\max\{X_1, Y_1\}, X_2\}$ and $U_2 = \min\{X_1, \max\{X_2, Y_1\}\}$, in which is observed that there is one spare allocation. Maybe for the first time (in our knowledge), Boland et al. (1992) showed that if $X_2 \geq_{st} X_1$ then $U_1 \geq_{st} U_2$, which reveals that, it is better to allocate the spare component in parallel with the weakest component, where $\geq_{st}$ denotes the usual stochastic order (for its formal definition please see Shaked and Shanthikumar (2007), Page 3). Then, Valdés and Zequeira (2003) extended the result in Boland et al. (1992) for the likelihood rate order (for its formal definition please see Shaked and Shanthikumar (2007), Page 16) to compare the lifetimes $U_1 = \min\{\max\{X_1, Y_1\}, X_2\}$ and $U_2 = \min\{X_1, \max\{X_2, Y_2\}\}$, in which is observed that there are two spare allocations, Y_1 and Y_2. After that for two spare allocations, Valdés and Zequeira (2006) gave conditions under which the allocation of the strongest spare with the weakest component is optimal in the sense of the hazard rate order.

Consider a general coherent systems ϕ with n components having independent lifetimes $X_1, \ldots, X_n$ and n active spares having independent lifetimes $Y_1, \ldots, Y_n$. Suppose that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are statistically independent. Recall that a system is said to be coherent system if it has no irrelevant components and the structure function of the system is monotone in each argument (that is, an improvement of a component cannot lead to a deterioration in system performance).

In this paper, we consider two scenario, matching spares and non-matching spares. In a matching spares problem, $X_i =_{st} Y_i$ for all $i = 1, \ldots, n$ where $=_{st}$ stands that the random variables have the same distribution, and in a non-matching spares problem $X_i \neq_{st} Y_i$ for all $i = 1, \ldots, n$. Denote $\tau(\mathbf{X}) = \tau(X_1, \ldots, X_n)$ the lifetime of the coherent system ϕ. When all components and spares lifetimes are independent and identically distributed, Boland and El-Neweihi (1995) proved, for general coherent systems, under some mild conditions, that $\tau(\mathbf{X} \vee \mathbf{Y}) \geq_{hr} \tau(\mathbf{X}) \vee \tau(\mathbf{Y}).wherethesymbols$‘$\wedge$’ and ‘$\vee$’ mean min and max, respectively. Noting that, they said that (1) can not be applied to the general k-out-of-n systems but they showed the result is true for the special case of 2-out-of-n system. After that, Singh and Singh (1997) extend (1) to a stronger result in the sense of the likelihood ratio order (for its formal definition please see Shaked and Shanthikumar (2007), Page 42) for the general k-out-of-n system. Gupta and Nanda (2001) obtained a similar result for the reversed hazard rate ordering. Also, there remains an open problem that whether this result still holds for the matching spares problem. Also, Misra, Dhariyal, and Gupta (2009) consider the non-matching problem and obtained some new results in this direction. Recently Zaho et al. (2015) studied stochastic comparisons of series systems at component level and system level with n exponential components under the set-up of matching spares problem.

In this paper, we focus our attention to the matching spares and non-matching spares and obtained some interesting results to compare the coherent systems with n exponential components and with n proportional reversed hazard rate ($PRHR$) components that redundancies as active spares at the system level is better than the system level in terms

of the hazard rate order and the reversed hazard rate order.

2 Main results

In this section, we discuss matching spares and non-matching spares, respectively. We obtain some results under which component redundancy to be superior to the system redundancy with respect to the hazard rate order and the reversed hazard rate order, for any n-component series system. Before presenting our main results, let us first introduce the $PRHR$ model. Independent random variables $X_1, \ldots, X_n$ are said to follow the $PRHR$ model if for $i = 1, \ldots, n$, the distribution function of X_i can be expressed as,

$$F_i(x) = F^{\lambda_i}(x),\ \lambda_i > 0,$$

where F is the distribution function of the base line distribution and λ_i is the parameter of the model.

Theorem 1. *Let $X_1, \ldots, X_n$ be independent random variables following the $PRHR$ model with distribution functions $F^{\lambda_1}, \ldots, F^{\lambda_n}$, respectively. Let $Y_1, \ldots, Y_n$ be another independent random variables following the $PRHR$ model with parameters $F^{\lambda_1}, \ldots, F^{\lambda_n}$, respectively. Then*

$$\wedge\{X_1 \vee Y_1, \ldots, X_n \vee Y_n\} \geq_{hr} \{\wedge(X_1, \ldots, X_n)\} \vee \{\wedge(Y_1, \ldots, Y_n)\}.$$

Theorem 2. *Let $X_1, \ldots, X_n$ be independent random variables following the $PRHR$ model with common distribution function F^{λ}. Let $Y_1, \ldots, Y_n$ be another independent random variables following the $PRHR$ model with common distribution function F^{β}. Then*

$$\wedge\{X_1 \vee Y_1, \ldots, X_n \vee Y_n\} \geq_{rh} \{\wedge(X_1, \ldots, X_n)\} \vee \{\wedge(Y_1, \ldots, Y_n)\},$$

where $\geq_{rh}$ is the reversed hazard rate order (for its formal definition please see Shaked and Shanthikumar (2007), Page 36).

Theorem 3. *Let $X_1, \ldots, X_n$ be independent random variables following the exponential distribution with common parameter λ. Let $Y_1, \ldots, Y_n$ be another independent random variables following the exponential distribution with common parameter β. Then*

$$\wedge\{X_1 \vee Y_1, \ldots, X_n \vee Y_n\} \geq_{hr} \{\wedge(X_1, \ldots, X_n)\} \vee \{\wedge(Y_1, \ldots, Y_n)\}.$$

References

[1] Boland, P. J., El-Neweihi, E., and Proschan, F. (1992). Stochastic order for redundancy allocation in series and parallel systems. *Advances in Applied Probability.* **24**, 161-171.

[2] Boland, P. J., and El-Neweihi, E. (1995). Component redundancy versus system redundancy in the hazard rate ordering. *IEEE Transactions on Reliability.* **44**, 614-619.

[3] da Costa Bueno, V., and do Carmo, I. M. (2007). Active redundancy allocation for a k-out-of-n:F system of dependent components. *European Journal of Operational Research.* **176**, 1041-1051.

[4] Laniado, H., and Lillo, R. E. (2014). Allocation policies of redundancies in two-parallel-series and two-series-parallel systems. *IEEE Transactions on Reliability.* **63**, 223-229.

[5] Singh, H., and Misra, H. (1994). On redundancy allocation in systems. *Journal of Applied Probability*. **31**, 1004-1014.

[6] Shaked, M., and Shanthikumar, J. G. (2007). Stochastic orders. New York: Springer-Verlag.

[7] Valdés, J.E., and Zequeira, R. (2003). On the optimal allocation of an active redundancy in a two-component series system. *Statistical & Probability Letters*. **63**, 325-332.

[8] Valdés, J.E., and Zequeira, R. (2006). On the optimal allocation of two redundancies in a two-component series system. *Operation Research Letter*. **34**, 49-52.

[9] Zhao, P., Chan, P., and Ng, H. K. T. (2012). Optimal allocation of redundancies in series systems. *European Journal of Operational Research* **220**, 673-683.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Prediction of Lower Records Based on Hybrid Censored Samples for the Topp-Leone Distribution

Aminzadeh, M. [1] and MirMostafaee, S.M.T.K. [2] and Safarpour, N. [3]

[1,2] Department of Statistics, Faculty of Mathematical Sciences, University of Mazandaran, P. O. Box 47416-1467, Babolsar, Iran

[3] Department of Statistics, Tarbiat Modares University, 14115-134; Tehran, Iran

Abstract

In this paper, we discuss maximum likelihood prediction of record values from the Topp-Leone distribution based on observed hybrid censored data. We also derive Bayes predictors for the future records under the squared error loss (SEL) function. A simulation study is presented for comparing the classical and Bayes methods in the end.

Keywords: hybrid censoring, maximum likelihood prediction, records.

1 Introduction

In many lifetime experiments, we encounter samples with censored units. [3] introduced Type I hybrid censoring scheme which can be described as follows: Consider a sample of n units with lifetimes $X_1, \cdots, X_n$ placed on a life-test at time 0 and suppose that $X_{1:n} < \ldots < X_{n:n}$ denote the ordered lifetimes. Then, the experiment will be ended at $T_0 = \min\{X_{r:n}, T\}$ where T is a pre-specified time. Thus the hybrid censored sample may be denoted as $\mathbf{X} = (X_{1:n}, \cdots, X_{d:n})$, where $X_{d:n} \leq T_0$, $d \leq r$ and $X_{d+1:n} \geq T_0$. For a random sample $X_1, \cdots, X_n$ coming from a distribution with probability density function

[1]m.aminzadeh1367@gmail.com
[2]m.mirmostafaee@umz.ac.ir
[3]niloufar.safarpour1367@gmail.com

(pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$, the joint pdf of the hybrid censored sample is given by (see for example [2])

$$f_{X_{1:n},\cdots,X_{d:n}}(x_1,\cdots,x_d) = C[1-F(T_0)]^{n-d}\prod_{i=1}^{d} f(x_i), \tag{1}$$

where $x_1 < \cdots < x_d$ and C is the normalizing constant.

Records are widely used in many real life applications like lifetime experiments. Let $f(x)$ and $F(x)$ be the pdf and cdf of X_1, respectively, then the pdf of the mth lower record value, denoted by U_m, is (see [1])

$$f_{U_m}(u) = \frac{[-\log F(u)]^{m-1}}{(m-1)!} f(u). \tag{2}$$

Assume that the underlying distribution is the Topp-Leone (TL) distribution with pdf

$$f(x;\theta) = 2\theta(1-x)(2x-x^2)^{\theta-1}, \quad 0<x<1, \quad \theta>0. \tag{3}$$

Let $\boldsymbol{X} = (X_{1:n},\cdots,X_{d:n})$ be a hybrid censored sample from the TL distribution with pdf (3) and U_m be the mth lower record extracted from a future sequence of random variables coming from the same distribution which is assumed to be independent from $\boldsymbol{X}$. The aim of this paper is to predict U_m based on the observed hybrid censored sample, $\boldsymbol{x} = (x_1,\cdots,x_d)$. Main results as well as a simulation study are presented in the next section.

2 Main results

First, we investigate how to find the maximum likelihood predictor (MLP) of U_m. The predictive likelihood function (PLF) approach was first applied by [4]. From (1) and (2), the joint pdf of U_m and $X_{1:n},\cdots,X_{d:n}$ is obtained to be

$$\begin{aligned} f_{U_m,X_{1:n},\cdots,X_{d:n}}(u,x_1,\cdots,x_d;\theta) &= \frac{2C\theta^{m+d}}{(m-1)!}[1-(2T_0-T_0^2)^{\theta}]^{n-d}[-\xi(u)]^{m-1} \\ &\times (1-u)e^{(\theta-1)[\xi(u)+\sum_{i=1}^{d}\xi(x_i)]}\prod_{i=1}^{d}(2-2x_i), \end{aligned}$$

where $\xi(x) = \log(2x-x^2)$. Denote the log-likelihood by ℓ, then we obtain (ignoring the constant)

$$\begin{aligned} \ell \propto\ & (d+m)\log\theta + (n-d)\log[1-(2T_0-T_0^2)^{\theta}] + (m-1)\log[-\xi(u)] \\ & + \log(1-u) + (\theta-1)\left[\xi(u)+\sum_{i=1}^{d}\xi(x_i)\right] + \sum_{i=1}^{d}\log(2-2x_i). \end{aligned}$$

Hence, the predictive likelihood equations are as follows

$$\begin{aligned} \frac{\partial\ell}{\partial\theta} &= \frac{d+m}{\theta} - \frac{(n-d)(2T_0-T_0^2)^{\theta}\xi(T_0)}{1-(2T_0-T_0^2)^{\theta}} + \xi(u) + \sum_{i=1}^{d}\xi(x_i) = 0, \\ \frac{\partial\ell}{\partial u} &= \frac{1}{u-1} - \frac{(m-1)(2-2u)}{(2u-u^2)\xi(u)} + (\theta-1)\frac{2-2u}{2u-u^2} = 0. \end{aligned}$$

We used the function `optim` in R 3.1.2 to find the MLP in our simulation study. Now, let us consider the gamma distribution as the prior distribution for the shape parameter θ

$$\pi(\theta) \propto \theta^{\alpha-1}\exp\{-\beta\theta\}, \quad \alpha,\beta,\theta>0. \tag{4}$$

From (1) and (2.3), the posterior distribution of θ given the hybrid censored sample, $\mathbf{x}=(x_1,\cdots,x_d)$, is found to be

$$\pi(\theta|\mathbf{x}) = \frac{1}{C_0}[1-(2T_0-T_0^2)^{\theta}]^{n-d}\theta^{\alpha+d-1}\exp\left\{-\theta\left(\beta-\sum_{i=1}^{d}\xi(x_i)\right)\right\},$$

where $C_0=\int_0^{\infty}[1-(2T_0-T_0^2)^{\theta}]^{n-d}\theta^{\alpha+d-1}e^{-\theta[\beta-\sum_{i=1}^{d}\xi(x_i)]}\mathrm{d}\theta$.

Therefore, by using the binomial expansion and after some algebra, the Bayes predictive density function of U_m is obtained to be

$$\begin{aligned} f^*_{U_m}(u|\mathbf{x}) &= \int_0^{\infty} f_{U_m}(u|\theta)\pi(\theta|\mathbf{x})\mathrm{d}\theta = \frac{2(1-u)[-\xi(u)]^{m-1}}{C_0(m-1)!(2u-u^2)} \\ &\quad\times\sum_{k=0}^{n-d}\frac{\binom{n-d}{k}(-1)^k\Gamma(m+\alpha+d)}{\left[\beta-\xi(u)-\sum_{i=1}^{d}\xi(x_i)-k\xi(T_0)\right]^{m+\alpha+d}}. \end{aligned}$$

Under the SEL function, the BPP of the U_m, denoted as $\widehat{U}_m^B$, will be the mean of the predictive density i.e.

$$\begin{aligned} \widehat{U}_m^B &= \int_0^1 u f^*_{U_m}(u|\mathbf{x})\mathrm{d}u = \frac{2\Gamma(m+\alpha+d)}{C_0(m-1)!}\sum_{k=0}^{n-d}\binom{n-d}{k}(-1)^k \\ &\quad\times\int_0^1\frac{u(1-u)[-\xi(u)]^{m-1}}{(2u-u^2)\left[\beta-\xi(u)-\sum_{i=1}^{d}\xi(x_i)-k\xi(T_0)\right]^{m+\alpha+d}}\mathrm{d}u. \end{aligned}$$

We applied the function `integrate` in R 3.1.2 to evaluate the above integral in our simulation study.

2.1 A simulation study

Here, first we generate 5 record values from the TL distribution with $\theta=2$, and assume these simulated values are the records which must be predicted. These records are

$$0.66591273,\ 0.24130283,\ 0.13715531,\ 0.07468685,\ 0.04588906.$$

Let us consider three cases for the prior distribution. **Case I**: Jeffreys' non-informative prior with $\alpha=\beta=0$. **Case II**: Informative prior with prior information $E(\theta)=2$ =true value, and $Var(\theta)=2$ and from (2.3), we obtain $\alpha=2$ and $\beta=1$. **Case III**: Informative prior with prior information $E(\theta)=2$ and $Var(\theta)=0.5$ and consequently from (2.3), we have $\alpha=8$ and $\beta=4$. Consider the following algorithm:
1. Generate $n=20$ i.i.d random variables $X_1,\cdots,X_n$ from TL distribution with $\theta=2$.
2. Sort the the random sample generated in Step 1 and for $r=17$ and $T=6$, find the values d and T_0.
3. Based on the values $(X_{1:n},\cdots,X_{d:n})=(x_1,\cdots,x_d)$, obtain the MLP and BPP of the mth record for $m=1,\cdots,5$.
4. Repeat Steps 1–3 $N=10000$ times and obtain the estimated mean squared prediction errors (EMSPEs) of $\widehat{U}_m^M$ and $\widehat{U}_m^B$ for $m=1,\cdots,5$.
The EMSPEs can be obtained from the following formulae:

$$\mathrm{EMSPE}(\widehat{U}_m^M)=\frac{1}{N}\sum_{i=1}^{N}(\hat{u}_m^M(i)-u_m)^2 \quad \& \quad \mathrm{EMSPE}(\widehat{U}_m^B)=\frac{1}{N}\sum_{i=1}^{N}(\hat{u}_m^B(i)-u_m)^2,$$

where $\hat{u}_m^M(i)$ and $\hat{u}_m^B(i)$ are the obtained MLP and BPP of U_m from Step 3 at time i, respectively, and u_m is the assumed real value of the mth record. From Table 1, we

Table 1: The EMSPEs of the predictors.

m	MLP	BPP Case I	BPP Case II	BPP Case III
1	0.066862331	0.042062992	0.041827132	0.041318827
2	0.015685658	0.003292021	0.002863069	0.002120252
3	0.009661575	0.003118115	0.002726269	0.002057912
4	0.003888659	0.002813793	0.002488697	0.001942897
5	0.001741999	0.001783184	0.001546211	0.001156455

observe that the EMSPEs of the Bayes predictors are smaller than the EMSPEs of the corresponding MLPs (except for one case) revealing the superiority of Bayes method. We also see that the EMSPEs are decreasing with respect to m. Moreover, Case III contains the smallest EMSPEs which is reasonable as Case III has the smallest prior variance and is the most informative case.

References

[1] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998), *Records*, John Wiley & Sons, New York.

[2] Balakrishnan, N. and Kundu, D. (2012), Hybrid censoring: models, inferential results and applications (with discussion), *Computational Statistics and Data Analysis*, **57**, 166-209.

[3] Epstein, B. (1954), Truncated life tests in the exponential case, *Annals of Mathematical Statistics*, **25**, 555-564.

[4] Kaminsky, K.S. and Rhodin, L.S. (1985), Maximum likelihood prediction, *Annals of the Institute of Statistical Mathematics*, **37**, 507-517.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Three-State Networks Subject to Shocks

Ashrafi, S. [1] and Zarezadeh, S. [2]

[1] Department of Statistics, University of Isfahan, Isfahan 81744, Iran
[2] Department of Statistics, Shiraz University, Shiraz 71454, Iran

Abstract

In this paper, we consider a three-state network that is subject to shocks. We assume that these shocks occur according to a counting process and in each shock, more than one link may fail. Under some assumptions, we introduce a new variant of two-dimensional signature that is called *two-dimensional t-signature*. A mixture representation is provided for the joint reliability function of the states lifetimes of the network. Also, some stochastic and dependent properties of the states lifetimes of the network are investigated.

Keywords: Counting process, two-dimensional t-signature, positively quadrant dependent, upper orthant order.

1 Introduction

The analysis of network reliability has been extensively studied in the last decades. In many situations of real life, more than two states are considered for the network and is named multi-state network. Recently, the reliability of such networks has been investigated by many researchers; see for example [2] and [3]. In this paper, we consider a three-state network consisting of n links. The states of the network are up state ($K = 2$), partial performance ($K = 1$) and down state ($K = 0$). Further, we assume that the links are subject to failure and nodes are completely reliable. The random variables T_1 and T, respectively, denote the times that the network enters into state $K = 1$ and $K = 0$. Gertsbakh and Shpungin [1] considered a three-state network and under the assumption

[1] s.ashrafi@sci.ui.ac.ir
[2] s.zarezadeh@shirazu.ac.ir

that at each time instant at most one link may fail, gave a mixture representation for the reliability function of T_1 and T. In the proposed representation, the mixing probabilities are the elements of signature matrix. However, there are situations in which at each time instant more than one link may fail. Zrezadeh et. al. [4] studied this case for the two-state networks.

In this paper, we assume that a three-state network is subject to shocks such that in each shock more than one link may fail. We introduce a new concept "two-dimensional t-signature" that is a combinatorial property of the network. Then, we obtain a mixture representation for the joint reliability function of T_1 and T. Some stochastic and dependent properties of T_1 and T are explored.

Before giving the main results of the paper, we need the following concepts which will be used throughout the paper.

Definition 1. (a) Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors with survival functions $\bar{F}$ and $\bar{G}$, respectively. $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the upper orthant order (denoted by $\mathbf{X} \leq_{uo} \mathbf{Y}$) if $\bar{F}(\mathbf{x}) \leq \bar{G}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R^n}$.

(b) The nonnegative function $f(x_1, x_2)$ is called totally positive of order 2 (TP_2) if for all $x_1 < y_1, x_2 < y_2$

$$f(x_1, x_2)f(y_1, y_2) \geq f(x_1, y_2)f(y_1, x_2).$$

(c) The random vector (X, Y) is said to be positively quadrant dependent (PQD) if $\bar{F}(x, y) \geq \bar{F}(x)\bar{F}(y)$.

2 Main Results

In this paper, we consider a three-state network that is subject to shocks that appear according to a counting process. In such a situation, this is possible that more than one link fail in each shock. In general, there are many real life situations in which more than one link may fail at each time instant i.e. there may be ties between the lifetimes of the components. Let $X_1, ..., X_n$ be i.i.d. random variables that show the lifetimes of the links of a three-state network. Gertsbakh and Spungin [1] defined the notion of two-dimensional signature under the assumption that there are not ties between $X_1, ..., X_n$ i.e. $P(X_i = X_j) = 0$ for every $i \neq j$. Suppose that in each shock may exist more than one link failure. In such a situation, we extend the notion of two-dimensional signature as follows.

Definition 2. Consider a three-state network with n links. Assume that all permutations of the links failures are equally likely. Suppose that the random variables M_1, M_2 and M_3 are the minimum number of links that their failures cause the state of the network changes, respectively, from $K = 2$ to $K = 1$, from $K = 1$ to $K = 0$ and from $K = 2$ to $K = 0$. The two dimensional t-signature of the network is defined as the matrix $S^\tau = (s^\tau_{i,j})$ where the non-zero elements are defined as

$$s^\tau_{i,j} = \frac{n_{i,j}}{n^*}, \quad 1 \leq i < j \leq n, \qquad s^\tau_{i,i} = \frac{n_i}{n^*}, \quad i = 1, ..., n,$$

where n^* is the number of ways that the links fail in the network, $n_{i,j}$ is the number of ways of links failures in which $M_1 = i$ and $M_2 = j$ and n_i is the number of ways of links failures in which $M_3 = i$.

Note the value of n^* is computed in Lemma 1 of [4]. Also, two-dimensional t-signature depends only on the structure of the network and does not depend on the links lifetimes.

Example 1. Consider the network pictured in Figure 1 with three terminals a, b and d. We define three states for the network. It is in state $K = 2$, if all terminals are connected, $K = 1$, if just two terminals are connected, and $K = 0$ if all terminals are disconnected. Using Lemma 1 of [4], it can be shown that $n^* = 75$. One can also verify that the nonzero elements of S^τ are given as $s^\tau_{2,3} = \frac{33}{75},\ s^\tau_{2,4} = \frac{20}{75},\ s^\tau_{3,4} = \frac{8}{75},\ s^\tau_{3,3} = \frac{13}{75},\ s^\tau_{4,4} = \frac{1}{75}$.

[node distance = 3 cm] LabelStyle/.style = semithick,fill= white,scale=0.8, text=black [scale=0.9,semithick,shape = circle,draw, fill= black, text= white, inner sep =1.3pt, outer sep= 0pt, minimum size= 3 pt](A) at (0,0) a; [scale=0.8,semithick,shape = circle,draw, fill= black, text= white, inner sep =1.3pt, outer sep= 0pt, minimum size= 3 pt](B) at (1,1) b; [scale=0.9,semithick,shape = circle,draw, fill= white, text= black, inner sep =1.3pt, outer sep= 0pt, minimum size= 3 pt](C) at (1,-1) c; [scale=0.8,semithick,shape = circle,draw, fill= black, text= white, inner sep =1.3pt, outer sep= 0pt, minimum size= 3 pt](D) at (2,0) d; [semithick] (A) to node[LabelStyle]1 (B); [semithick] (A) to node[LabelStyle]2 (C); [semithick] (B) to node[LabelStyle]3 (D); [semithick] (C) to node[LabelStyle]4 (D);

Figure 1: Network with 4 nodes and 4 links.

In the following, we consider a three-state network that is subject to shocks. Suppose that shocks appear according to a counting process, denoted by $\{\xi(t), t > 0\}$, at random time instants $\vartheta_1, \vartheta_2, \ldots$. Let random variable W_i, $i = 1, 2, \ldots$, denotes the number of links that fail at the ith shock and $H_{k,l}(x, y)$ denote the joint distribution function of r.v.'s $\sum_{i=0}^{k} W_i$ and $\sum_{i=0}^{l} W_i$.

Suppose that $\beta_{k,l} = \sum_{i=1}^{n} \sum_{j=i}^{n} s^\tau_{i,j} H_{k,l}(i-1, j-1)$ and $b_{k,l} = \beta_{k-1,l-1} - \beta_{k,l-1} - \beta_{k-1,l} + \beta_{k,l}$. Under the assumption that the process of occurrence of the shocks is independent of the number of failed links, and the fact that the total number of components that fail up to time t is independent of the two-dimensional t-signature, we obtain the joint reliability function of T_1 and T as

$$P(T_1 > t_1, T > t) = \sum_{k=0}^{\infty} \sum_{l=k}^{\infty} b_{k,l} P(\vartheta_k > t_1, \vartheta_l > t), \quad t_1 < t.$$

In the following theorem, we compare two networks based on their structures.

Theorem 1. *Consider two networks each consists of n links having t-signature matrices S^τ_1 and S^τ_2, respectively. Let the links of ith network are subject to shocks that appear according to counting process $\{\xi_i(t), t \geq 0\}$, $i = 1, 2$ with epoch times $\vartheta_{i,1}, \vartheta_{i,2}, \ldots$. Suppose that $W_{i,j}$, $i = 1, 2$, $j = 1, 2, \ldots$ shows the number of failed links of ith network at jth shocks. If $(\vartheta_{1,i}, \vartheta_{1,j}) \leq_{uo} (\vartheta_{2,i}, \vartheta_{2,j})$ for $1 \leq i < j \leq n$, $S^\tau_1 \leq_{uo} S^\tau_2$ and $(\sum_{i=1}^{k} W_{1,i}, \sum_{i=1}^{l} W_{1,i}) \geq_{uo} (\sum_{i=1}^{k} W_{2,i}, \sum_{i=1}^{l} W_{2,i})$ for all $k \leq l$ then $(T_1^{(1)}, T^{(1)}) \leq_{uo} (T_1^{(2)}, T^{(2)})$.*

In the following theorem, positive dependence relationship between T_1 and T is explored. Before it, denote $\bar{S}^\tau_{i,j} = \sum_{k=i+1}^{n-1} \sum_{j=\max\{k,j+1\}}^{n-1} s^\tau_{k,l}$.

Theorem 2. *Let T_1 be the lifetime of a network in state $K = 2$ and T be the lifetime of the network. Let the network is subject to shocks that appear according to a counting processes $\{\xi(t), t \geq 0\}$ with epoch times $\vartheta_1, \vartheta_2, \ldots$. If $\vartheta_1, \vartheta_2, \ldots$ form a markov chain with TP_2 transition probability densities, $H_{k,l}(i-1, j-1) \geq H_k(i-1) H_l(j-1)$ and $\bar{S}^\tau_{i,j} \geq \bar{S}^{(1)\tau}_i \bar{S}^{(2)\tau}_j$ for each $k \leq l$ and $1 \leq i \leq j \leq n$ then T_1 and T are PQD.*

References

[1] Gertsbakh, I. and Shpungin, Y. (2011), *Network Reliability and Resilience*, Springer Briefs, Springer.

[2] Li, W. and Zuo, M.J. (2008), Reliability evaluation of multi-state weighted k-out-of-n systems, *Reliability Engineering & System Safety*, **93(1)**, 160-167.

[3] Lisnianski, A. and Levitin, G. (2003), *Multi-state System Reliability: Assessment, Optimization and Applications*. World Scientific, Singapore.

[4] Zarezadeh, S., Ashrafi, S. and Asadi, M. (2015), A Shock Model Based Approach to Network Reliability. *Reliability, IEEE Transactions on*, doi 10.1109/TR.2015.2494366.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

The mean residual life and its connection to information measures

Majid Asadi [1]

[1] Department of Statistics, University of Isfahan, Isfahan, 81744, IRAN

Abstract

The mean residual life (MRL) of a nonnegative random variable X plays an important role in reliability theory, survival analysis and other branches of probability and statistics. In this talk, we address the connection between the MRL and some uncertainty measures. First, we show that the MRL has a close relation to the variance of X and as an application, it is shown that the exponential distribution has maximum cumulative residual entropy (CRE) among the distributions with given standard deviation. Then, we define a Tsallis entropy version of the CRE and show that it is closely connected to the concept of MRL. Several properties and applications of the proposed measure in reliability engineering will be also discussed.

Keywords: Hazard rate, Gini's coefficient, Residual entropy, Maximum entropy, Tsallis entropy. Coherent systems.

[1] asadi4@hotmail.com

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Bayesian Estimation for the Marshall-Olkin Extended Exponential Distribution Under Progressively Type-II Censoring with Binomial Removals

Azizi, E. [1] and MirMostafaee, S.M.T.K. [2]

[1,2] Department of Statistics, University of Mazandaran, P. O. Box 47416-1467, Babolsar, Iran

Abstract

In this paper, we study the Bayesian estimation problem for the Marshall-Olkin extended exponential distribution under Type-II progressive censoring scheme with binomial removals. In the end, a simulation study is presented to illustrate the proposed procedures.

Keywords: Gibbs sampling, Bayes estimator, binomial removals.

1 Introduction

The Marshall-Olkin extended exponential (MOEE) distribution was first introduced by [3], whose probability density function (pdf) is

$$f(x|\theta) = \theta e^{-x}[1-(1-\theta)e^{-x}]^{-2}, \quad x > 0, \quad \theta > 0. \tag{1}$$

In what follows, we focus on Bayesian estimation for this model under Type-II progressive censoring with binomial removals (Type-II PCBR).

[1]esmaeelazizi1369@yahoo.com
[2]m.mirmostafaee@umz.ac.ir

2 Main results

Let $\boldsymbol{X} = (X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n})$ be a progressively Type-II right censored sample of size m extracted from a sample of size n, where lifetimes have an MOEE distribution with pdf given in (1). At the i-th failure, $R_i,\ i = 1, ..., m-1$ units are removed randomly from the experiment and when the m-th failure is observed, the remaining $n - m - \sum_{j=1}^{m-1} R_j$ units are all removed. Suppose that R_i's are discrete random variables such that R_1 follows $Bin(n-m, p)$ and $R_i|R_1 = r_1, ..., R_{i-1} = r_{i-1}$ follows $Bin(n - m - \sum_{j=1}^{i-1} r_j, p)$ for $i = 2, ..., m-1$, where $Bin(n, p)$ denotes the binomial distribution with parameters n and p. Assume $\boldsymbol{R} = (R_1, ..., R_{m-1})$ and $\boldsymbol{X}$ are independent. Then the likelihood function given $\boldsymbol{X} = \boldsymbol{x}$ and $\boldsymbol{R} = \boldsymbol{r}$ is (see for example [2])

$$L(\theta, p) = \frac{C(\boldsymbol{r})(n-m)! e^{-\sum_{i=1}^{m} x_i(1+r_i)} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\theta^{-n} \prod_{i=1}^{m} [1-(1-\theta)\exp\{-x_i\}]^{r_i+2} \prod_{i=1}^{m-1} r_i! (n-m-\sum_{i=1}^{m-1} r_i)!},$$

where $C(\boldsymbol{r}) = n(n-r_1-1)\cdots(n-r_1-r_2-\cdots-r_{m-1}-m+1)$ and $r_m = n - m - \sum_{j=1}^{m-1} r_j$. It can be easily found out that the MLE of p, denoted as $\widehat{p}_M$, is $\widehat{p}_M = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i-1)r_i}$, and the MLE of θ, denoted as $\widehat{\theta}_M$, is the solution of the following equation

$$\frac{n}{\theta} - \sum_{i=1}^{m} \frac{(r_i+2)e^{-x_i}}{1-(1-\theta)e^{-x_i}} = 0.$$

For Bayesian estimation, we propose independent priors for the parameters θ and p as $\pi_1(\theta) \propto \theta^{a-1} e^{-b\theta}$, and $\pi_2(p) \propto p^{c-1}(1-p)^{d-1}$, respectively, where $\theta > 0$, $0 < p < 1$ and a, b, c and d are positive hyperparameters. Thus the joint prior distribution for θ and p is

$$\pi(\theta, p) \propto \theta^{a-1} e^{-b\theta} p^{c-1} (1-p)^{d-1}.$$

The joint posterior distribution of θ and p is then obtained as

$$\pi(\theta, p|\boldsymbol{x}, \boldsymbol{r}) = \frac{p^{\alpha-1}(1-p)^{\beta-1}}{B(\alpha,\beta)C_0} \theta^{n+a-1} e^{-b\theta} \prod_{i=1}^{m} [1-(1-\theta)e^{-x_i}]^{-(r_i+2)},$$

where $\alpha = c + \sum_{i=1}^{m-1} r_i$, $\beta = (m-1)(n-m) + d - \sum_{i=1}^{m-1}(m-i)r_i$ and $C_0 = \int_0^\infty \theta^{n+a-1} e^{-b\theta} \prod_{i=1}^{m} [1-(1-\theta)e^{-x_i}]^{-(r_i+2)} \mathrm{d}\theta$. Therefore the marginal posterior (MP) pdf of θ is

$$\pi^*(\theta|\boldsymbol{x}, \boldsymbol{r}) = \frac{1}{C_0} \theta^{n+a-1} e^{-b\theta} \prod_{i=1}^{m} [1-(1-\theta)e^{-x_i}]^{-(r_i+2)}$$

and the MP pdf of p is beta distribution with parameters α and β. The Bayes estimate of θ under SEL function based on a Type-II PCBR sample, denoted as $\widehat{\theta}_S$, is the mean of the posterior density, i.e.

$$\widehat{\theta}_S = \frac{1}{C_0} \int_0^\infty \theta^{n+a} e^{-b\theta} \prod_{i=1}^{m} [1-(1-\theta)e^{-x_i}]^{-(r_i+2)} \mathrm{d}\theta \tag{2}$$

and the Bayes estimator of p under the SEL function is $\widehat{p}_S = \frac{\alpha}{\alpha+\beta}$. The SEL function may not be appropriate in situations that overestimation and underestimation have different

consequences as it is symmetric. Here, we consider the general entropy loss (GEL) function, defined as

$$L_2(\theta,\widehat{\theta}) = (\widehat{\theta}/\theta)^q - q\ln(\widehat{\theta}/\theta) - 1$$

The GEL function is asymmetric and positive values of q indicate that the overestimation is more serious than underestimation and vice versa. The Bayes estimate of θ under GEL function based on a Type-II PCBR sample, denoted as $\widehat{\theta}_G$, is $\widehat{\theta}_G = [E(\theta^{-q}|\boldsymbol{x},\boldsymbol{r})]^{-\frac{1}{q}}$, where

$$E(\theta^{-q}|\boldsymbol{x},\boldsymbol{r}) = \frac{1}{C_0}\int_0^\infty \theta^{n+a-1-q}e^{-b\theta}\prod_{i=1}^m[1-(1-\theta)e^{-x_i}]^{-(r_i+2)}\mathrm{d}\theta, \tag{3}$$

and the Bayes estimator of p under the GEL function is $\widehat{p}_G = [\frac{\Gamma(\alpha+\beta)\Gamma(\alpha-q)}{\Gamma(\alpha)\Gamma(\alpha+\beta-q)}]^{-1/q}$.

The integrals given in (2) and (3) may not be expressed in closed forms. Therefore, we apply the Markov chain Monte Carlo (MCMC) method and the Gibbs sampler to simulate samples from the posterior density and then approximate the Bayes estimators of θ. The posterior density function of θ can be rewritten as $\pi^*(\theta|\boldsymbol{x},\boldsymbol{r}) \propto g^*(\theta|\boldsymbol{x},\boldsymbol{r})h^*(\theta)$, where $g^*(\theta|\boldsymbol{x},\boldsymbol{r})$ is the gamma density function with parameters $n+a$ and b and $h^*(\theta) = \prod_{i=1}^m[1-(1-\theta)e^{-x_i}]^{-(r_i+2)}$. Now, consider the following algorithm to approximate the Bayes estimators

Step 1: Generate θ_1 from $g^*(\theta|\boldsymbol{x},\boldsymbol{r})$.
Step 2: Repeat Step 1, N times to obtain $\theta_1,\cdots,\theta_N$.
Step 3: The approximate values for $\widehat{\theta}_S$ and $\widehat{\theta}_G$ are $\widehat{\theta}_{MS} = \sum_{i=1}^N \theta_i w_i$ and $\widehat{\theta}_{MG} = \left(\sum_{i=1}^N \theta_i^{-q} w_i\right)^{-\frac{1}{q}}$, respectively, where $w_i = \frac{h^*(\theta_i)}{\sum_{j=1}^N h^*(\theta_j)}$.

2.1 A simulation study

In this section, we performed a simulation to illustrate the proposed procedures. The following algorithm can be used to generate Type-II PCBR samples from the MOEE distribution.

Step 1: Fix the values of $\theta,\ p,\ n$ and m.
Step 2: Generate a random number, R_1, from $Bin(n-m,p)$ and $R_i|R_1=r_1,...,R_{i-1}=r_{i-1}$ from $Bin(n-m-\sum_{j=1}^{i-1}r_j,p)$ for $i=2,...,m-1$ and set $r_m = n-m-\sum_{k=1}^{m-1}r_k$.
Step 3: Given $\boldsymbol{R}=\boldsymbol{r}$, generate a progressively Type-II censored sample from the standard uniform distribution using the algorithm given in [1], page 32. Then the desired Type-II PCBR sample, $(X_{1:m:n},\cdots,X_{m:m:n})$, can be obtained by setting $X_{i:m:n} = \log\left([1-(1-\theta)U_{i:m:n}]/[1-U_{i:m:n}]\right)$ for $i=1,\cdots,m$.

We took $n=20,\ 30$ with different values of $m(\leq n)$ in this simulation. We randomly generated $M=10000$ Type-II PCBR censored data from the MOEE distribution with $\theta=4$ and $p=0.5$. We then obtained MLEs in each iteration. For Bayesian estimation, we considered the prior with $a=32$ and $b=8$ for θ and the uniform prior with $c=d=1$ for p. The approximate Bayes estimators of θ under SEL and GEL (for $q=-0.5,\ 0.5$) functions, which are denoted by $\widehat{\theta}_{MS}(i)$ and $\widehat{\theta}_{MG}(i)$, in the i-th iteration, respectively, were obtained using the MCMC sampling procedure with $N=1000$. The estimated risks (ERs) of the estimators were calculated using the relations $ER_S(\widehat{\theta}_{MS}) = \frac{1}{M}\sum_{i=1}^M[\widehat{\theta}_{MS}(i)-\alpha]^2$, and $ER_G(\widehat{\theta}_{MG}) = \frac{1}{M}\sum_{i=1}^M[(\frac{\widehat{\theta}_{MG}(i)}{\theta})^q - q\ln(\frac{\widehat{\theta}_{MG}(i)}{\theta})-1]$. Actually, we computed the ERs of each Bayes estimator according to its own loss function. For the MLEs, we computed both ERs. Similarly, we computed the ERs of the Bayes and maximum likelihood estimators

of p. From Table 1, we observe that the Bayes estimators are superior to the MLEs. Moreover, when n is fixed, the ERs of estimators of θ are decreasing with respect to m but the ERs of estimators of p are increasing with respect to m.

Table 1: The results of the simulation.

(n,m)		ERs of the estimators of θ			ERs of the estimators of p		
		ER_S	ER_G		ER_S	ER_G	
			$q=-0.5$	$q=0.5$		$q=-0.5$	$q=0.5$
(20,10)	MLE	>1000	0.04830	32.9368	0.01435	0.00628	0.00638
	Bayes	0.23454	0.00154	0.00150	0.01135	0.00528	0.00550
(20,15)	MLE	4.04444	0.02359	0.02353	0.03052	0.01177	0.01222
	Bayes	0.23380	0.00152	0.00149	0.01848	0.00834	0.00899
(30,15)	MLE	>1000	0.02489	4.20776	0.00907	0.00416	0.00420
	Bayes	1.42751	0.00796	0.00863	0.00781	0.00370	0.00380
(30,20)	MLE	2.80609	0.01809	0.01788	0.01382	0.00595	0.00608
	Bayes	1.42170	0.00794	0.00861	0.01092	0.00499	0.00521
(30,25)	MLE	2.26150	0.01492	0.01482	0.03008	0.01185	0.01223
	Bayes	1.40676	0.00786	0.00853	0.01834	0.00845	0.00910

References

[1] Balakrishnan, N. and Aggarwala, R. (2000), *Progressive Censoring: Theory, Methods, and Applications*, Birkhäuser, Boston.

[2] Dey, S. and Dey, T. (2014), Statistical inference for the Rayleigh distribution under progressively Type-II censoring with binomial removal, *Applied Mathematical Modelling*, **38**, 974-982.

[3] Marshall A.W. and Olkin, I. (1997), A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, *Biometrika*, **84**, 641-652.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Statistical Inference of Accelerated Life Tests with Masked Interval Data under Tampered Failure Rate Model

Azizi, F. [1] and Torabi, L. [2]

[1,2] Department of Mathematics, Statistics and Computer Science, University of Tehran, Iran

Abstract

In this work, we consider the statistical analysis for step-stress accelerated life test in the presence of competing risks, where a product or an item is exposed to multiple risks and a failure can be caused by one of the multiple risks. Due to cost consideration or environmental restrictions, the cause of failure for an item is not exactly known. Then, masked cause of failure might occur in the collected data. The competing risks distribution is assumed to follow the Weibull distribution with common shape parameter. Interval cencoring is allowed and tampered failure rate model is hold. We apply the maximum likelihood approach via the expectation-maximization algorithm and use the parametric bootstrap method for constructing confidence interval of the unknown parameters. Finally, the precision of the estimates is assessed through Monte Carlo simulations and bootstrap confidence intervals are derived.

Keywords: Accelerated Life Tests, Expectation-Maximization Algorithm, Interval Censoring, Masked Data, Tampered Failure Rate.

1 Introduction

Today's the modern products are designed to operate without failure for years, decades or longer. Thus, traditional life tests are not suitable to collect information for estimating the failure time distribution. In such situations, Accelerate Life Tests(ALTs) are widely used to shorten the failure time of products by running them at higher than normal level of stress. The accelereted tests may be performed using constant stress or increasing stress

[1] fariba.azizi@khayam.ut.ac.ir
[2]

levels. A special class of the ALT is called the step-stress testing. Generally, information from step-stress test is extrapolated to obtain the estimates of the failure time distribution at normal condition through a model such as Tampered Failure Rate(TFR) model [1]. This model describs the effect of changing the stress on the failure rate. In this work, we consider a simple step-stress model in the presence of competing risks and interval censoring. We assumed that TFR is hold and the competing risks follow the Weibull distribution with common shape parameter. We also assumed that for some items, the cause of failure is known only to belong to a subset of the set of all possible causes. The cause is reported as masked. We apply the maximum likelihood approach via the expectation-maximization algorithm and use the parametric bootstrap method for constructing confidence interval of the unknown parameters. Some ralated work in the literature in ALT with masked data are [2], [3], [5] and [4].

2 Assumptions and model

Consider a simple step-stress ALT in which an item is subjected to the stress s_1 untill a fixed time τ_1. If it does not fail, the stress is increased to s_2 at time τ_1 and held constant till specified time τ_2. TFR model is defined as

$$\lambda(t) = \begin{cases} \lambda_1(t) & \text{if} \quad 0 < t \leq \tau_1 \\ \alpha\lambda_1(t) & \text{if} \quad \tau_1 < t \leq \tau_2. \end{cases}$$

Then the CDF of the test unit is expressed as

$$F(t) = \begin{cases} F_1(t) & \text{if} \quad 0 < t \leq \tau_1 \\ 1 - (1 - F_1(\tau_1))^{1-\alpha}(1 - F_1(t))^{\alpha} & \text{if} \quad \tau_1 < t \leq \tau_2. \end{cases}$$

Let T_j, $j = 1, 2, ..., s$ be the latent failure time corresponding to j-th cause of failure for a test unit and $T = \min(T_1, T_2, ..., T_s)$ be the failure time of a test unit. From the assuption that the model is TFR, the CDF of T_j is given by

$$F_j(t) = \begin{cases} 1 - e^{-\lambda_j t^{\delta}} & \text{if} \quad 0 < t \leq \tau_1 \\ 1 - e^{-\lambda_j {\tau_1}^{\delta} - \alpha_j \lambda_j (t^{\delta} - {\tau_1}^{\delta})} & \text{if} \quad \tau_1 < t \leq \tau_2. \end{cases}$$

Setting $\theta_j = \alpha_j \lambda_j$ and

$$C = \begin{cases} 1, & T = T_1 \\ 2, & T = T_2 \\ . & . \\ . & . \\ . & . \\ s, & T = T_s, \end{cases}$$

the joint CDF of T and C are given by

$$F_{T,C}(t, j) = \begin{cases} \frac{\lambda_j}{\lambda^{\star}}(1 - e^{-\lambda^{\star} t^{\delta}}) & \text{if} \quad 0 < t \leq \tau_1 \\ \frac{\theta_j}{\theta^{\star}}(1 - e^{-\lambda^{\star} {\tau_1}^{\delta} - \theta^{\star}(t^{\delta} - {\tau_1}^{\delta})}) & \text{if} \quad \tau_1 < t \leq \tau_2, \end{cases} \quad (5)$$

where $\lambda^\star = \Sigma_{j=1}^{s}\lambda_j$ and $\theta^\star = \Sigma_{j=1}^{s}\theta_j$. As a result of masking, in addition to the model parameters λ_j, θ_j and δ, one must consider the masking probabilities for $k = 1, 2$ and $i = 1, 2, ..., n_k$ as

$$P_j(m_{ik}) = P(M_{ik} = m_{ik}|\tau_{k-1} < T_i \leq \tau_k, C_i = j), \qquad j\epsilon m_{ik},$$

that is the probability when item i fails in the interval $(\tau_{k-1} < t \leq \tau_k]$, due to cause j but it is masked by m_{ik}. Note that $\sum_{m_{ik}:j\epsilon m_{ik}} P_j(m_{ik}) = 1$. For simplicity we consider the special case that the only possible masking set is $\{1, 2, ..., s\}$. Assuming time-independent masking probabilities, we have

$$P(M_{ik} = m_{ik}|C_i = j) = \begin{cases} p_j & \text{if} \quad m_{ik} = \{j\} \\ 1 - p_j & \text{if} \quad m_{ik} = \{1, 2, ..., s\} \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

where $P_j = P(m_{ik} = \{j\}\,|C_i = j)$ for all $j = 1, 2, ..., s$.

3 Maximum Likelihood Estimation and EM Algorithm

Let M_k and n_{jk} denote the set of items failed and masked in the interval $(\tau_{k-1}, \tau_k]$ and the number of items failed in this interval due to cause j where $\tau_0 = 0$. Then the log-likelihood function under compelet data is

$$\begin{aligned} l_C \propto & \sum_{j=1}^{I}\Bigg\{\Bigg(\sum_{k=1}^{2} n_{jk}\Bigg)\log p_j + \Bigg(\sum_{k=1}^{2}\sum_{i\epsilon M_k} I_{ijk}\Bigg)\log(1-p_j) \\ & + \Bigg(n_{j1} + \sum_{i\epsilon M_1} I_{ij1}\Bigg)\Bigg(\log\frac{\lambda_j}{\lambda^\star} + \log(1 - e^{\lambda^*\tau_1^\delta})\Bigg) \\ & + \Bigg(n_{j2} + \sum_{i\epsilon M_2} I_{ij2}\Bigg)\Bigg(\log\frac{\theta_j}{\theta^\star} + \log(1 - e^{\theta^*(\tau_2^\delta - \tau_1^\delta)}) - \lambda^*\tau_1^\delta\Bigg)\Bigg\} \\ & - (r_1 + r_2)\lambda^*\tau_1^\delta - r_2\theta^*(\tau_2^\delta - \tau_1^\delta). \end{aligned} \quad (7)$$

where

$$I_{ijk} = \begin{cases} 1 & \text{if} \quad \text{item i fails in the interval } (\tau_{k-1}, \tau_k] \text{ due to cause j} \\ 0 & \text{if} \quad \text{otherwise,} \end{cases}$$

for all $j = 1, 2, ..., s$, $k = 1, 2$, $i = 1, 2, ..., n_k$. The use of EM to maximize (7) is recommended since the log-likelihood contains the missing data $\{I_{ijk}, i\epsilon M_k, k = 1, 2, j = 1, 2, ..., s\}$. Let $\lambda = (\lambda_1, \lambda_2, ...\lambda_s)$, $\theta = (\theta_1, \theta_2, ..., \theta_s)$, $p = (p_1, p_2, ..., p_s)$. For each $i\epsilon M_k$, we have

$$E(I_{ijk}|\lambda.\theta, p, \delta) = \begin{cases} \dfrac{(1-p_j)\lambda_j}{\Sigma_{j'\epsilon M_{i1}}(1-p_{j'})\lambda_{j'}} & \text{if } M_{i1} = \{1, 2, ..., s\} \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

and for each $i\epsilon M_2$, we have

$$E(I_{ijk}|\lambda.\theta,p,\delta)=\begin{cases}\dfrac{(1-p_j)\theta_j}{\Sigma_{j'\epsilon M_{i2}}(1-p_{j'})\theta_{j'}} & \text{if } M_{i2}=\{1,2,...,s\}\\ 0 & \text{otherwise,}\end{cases} \quad (10)$$

To obtain the maximum likelihood estimators of λ, θ, p and δ, we maximize (7) in which the missing I_{ijk} are substituted by (9) and (10).

References

[1] Bhattacharyya, G. K. and Soejoeti, Z. (1989). A tampered failure rate model for step-stress accelerated life test, Communications in statistics-theory and methods, 18(5): 1627-1643.

[2] Fan, T. H. and Wang, W. L. (2011). Accelerated Life Tests for Weibull Series Systems With Masked Data, *IEEE TRANSACTIONS ON RELIABILITY*, **60**, 557-569.

[3] Fan, T. H. and Hsu, T. M. (2012). Accelerated Life Tests of a Series System With Masked Interval Data Under Exponential Lifetime Distributions, *IEEE TRANSACTIONS ON RELIABILITY*, **61**, No. 3.

[4] Xu, A., Basu, S. and Tang, Y. (2014). A Full Bayesian Approach for Masked Data in Step-Stress Accelerated Life Testing, *IEEE TRANSACTIONS ON RELIABILITY*, **63**, 798-806.

[5] Xu, A., Tang, Y. and Guan, Q. (2014). Bayesian Analysis of Masked Data in Step-Stress Accelerated Life Testing, *Communications in Statistics- Simulation and Computation*, **43**, 2016-2030.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Inference of $R = P[X < Y]$ for proportional reversed hazard rate model

Salman Babayi [1] and Esmaeile Khorram [2]

[1,2] Department of Statistics, Faculty of Mathematics and Computer Science
Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Ave., Tehran, Iran

Abstract

The proportional reversed hazard rate model (PRHRM) has been extensively used in the literature to model failure time data. This paper deals with the estimation of $R = P[X < Y]$ when X and Y are two independent the PRHRM with different parameters. The maximum likelihood estimator of R is proposed. Assuming that the common scale parameter is known, the maximum likelihood estimator, uniformly minimum variance unbiased estimator, Bayes estimation and confidence interval of R are obtained. Finally, Analysis of a real data set has also been presented for illustrative purposes.

Keywords: Proportional reversed hazard rate model, Maximum likelihood estimator, Bayes estimator, Generalized logistic distribution.

1 Introduction

The problem of making inference about $R = P[X < Y]$ has received considerable attention in literature. This problem arises naturally in the context of mechanical reliability of a system with stress X and strength Y. The system fails, if at any time the applied stress is greater than its strength.

Various versions of this problem have been discussed in literature.

The concept of hazard rate is very well known in the reliability literature. To motivate the concept of reverse hazard rate, suppose the lifetime of a unit has reversed hazard rate $r(t) = (d/dt)lnF(t)$, where $F(t)$ is the distribution function.Then $r(t)$ dt is the conditional probability that the unit failed in an infinitesimal interval of width dt preceding t, given

[1]s.babayi@aut.ac.ir
[2]eskhor@aut.ac.ir

that it failed before t. In forensic science and in actuarial science, the time elapsed since failure is a quantity of interest in order to predict the actual time of failure. In this case $r(t)dt$ provides the probability of failing in $(t-dt,t)$, when a unit is found failed at time t.

Definition 1. *Let X is an absolutely continuous random variable with distribution function $F(.)$ and reversed hazard function $r(.)$. The family of random variables with reversed hazard function of the form $\{\alpha r(.) : \alpha > 0\}$ is called the proportional reversed hazard rate family and the distribution function $F(.)$ is called the baseline distribution function of that family.*

Therefore, a one-dimensional PRHRM is a parametric family of the distribution function:

$$F_{PRHRM}(x;\alpha,\sigma) = (F(x;\sigma))^{\alpha}; \qquad x \in S_X. \tag{1}$$

with parameters $\alpha > 0$, and θ and baseline distribution function $F(.;\sigma)$. The probability density function corresponding to the (1) is

$$f_{PRHM}(x;\alpha,\sigma) = \alpha(F(x;\sigma))^{\alpha-1} f(x;\sigma); \qquad x \in S_X.$$

Several members of the PRHRM have been found to be practical and flexible for analyzing real data, specially in the presence of censoring. Examples of such distributions contain generalized Logistic type I family, exponentiated Weibull family, generalized exponential family, the power normal family, exponentiated Rayleigh family and generalized exponential geometric family.

2 Maximum likelihood estimator of R with common scale parameter

In this section, we investigate the properties of R, when the common scale parameter σ, is the same.

Let $X \sim PRHRM(\alpha,\sigma)$ and $Y \sim PRHRM(\beta,\sigma)$, where X and Y are independent random variables. Therefore,

$$R = P(X < Y) = \frac{\beta}{\alpha+\beta}$$

So, in order to obtain the MLE of R, we need to compute the MLE of α and β. Suppose $X_1, X_2, ..., X_n$ is a random sample from $PRHRM(\alpha,\sigma)$ and $Y_1, Y_2, ..., Y_m$ is also a random sample from $PRHRM(\beta,\sigma)$. Therefore, the log-likelihood function of the observed sample is

$$\begin{aligned} L(\alpha,\beta,\sigma) =& n\ln\alpha + m\ln\beta - (n+m)\ln\sigma + \sum_{i=1}^{n}\ln f(\frac{x_i}{\sigma}) + \sum_{i=1}^{m}\ln f(\frac{y_i}{\sigma}) \\ &+ (\alpha-1)\sum_{i=1}^{n}\ln F(\frac{x_i}{\sigma}) + (\beta-1)\sum_{i=1}^{m}\ln F(\frac{y_i}{\sigma}). \end{aligned} \tag{2}$$

The MLE's of α and β say $\hat{\alpha}$ and $\hat{\beta}$ respectively, can be achived as follow:

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n} \ln F(\frac{X_i}{\hat{\sigma}})} \tag{3}$$

$$\hat{\beta} = \frac{-m}{\sum_{i=1}^{m} \ln F(\frac{Y_i}{\hat{\sigma}})} \tag{4}$$

where $\hat{\sigma}$ can be found by using an iterative scheme as follows:

$$h(\sigma_{(j)}) = \sigma_{(j+1)},$$

where

$$h(\sigma) = -(n+m)^{-1}[\sum_{i=1}^{n} \frac{x_i f'(\frac{x_i}{\sigma})}{f(\frac{x_i}{\sigma})} + \sum_{i=1}^{m} \frac{y_i f'(\frac{y_i}{\sigma})}{f(\frac{y_i}{\sigma})} + (\frac{-n}{\sum_{i=1}^{n} \ln F(\frac{x_i}{\sigma})} - 1) \sum_{i=1}^{n} \frac{x_i f(\frac{x_i}{\sigma})}{F(\frac{x_i}{\sigma})}$$

$$+ (\frac{-m}{\sum_{i=1}^{m} \ln F(\frac{y_i}{\sigma})} - 1) \sum_{i=1}^{m} \frac{y_i f(\frac{y_i}{\sigma})}{F(\frac{y_i}{\sigma})}]. \tag{5}$$

Since ML estimators are invariant, so the MLE of R becomes

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} \tag{6}$$

3 Estimation of R if σ is known

The real data sets.

Data Set I (x)							Data Set II (y)						
1.901	2.132	2.203	2.228	2.257	2.350	2.361	1.312	1.314	1.479	1.552	1.700	1.803	1.861
2.396	2.397	2.445	2.454	2.474	2.518	2.522	1.865	1.944	1.958	1.966	1.997	2.006	2.021
2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.027	2.055	2.063	2.098	2.140	2.179	2.224
2.659	2.675	2.738	2.740	2.856	2.917	2.928	2.240	2.253	2.270	2.272	2.274	2.301	2.301
2.937	2.937	2.977	2.996	3.030	3.125	3.139	2.359	2.382	2.382	2.426	2.434	2.435	2.478
3.145	3.220	3.223	3.235	3.243	3.264	3.272	2.490	2.511	2.514	2.535	2.554	2.566	2.570
3.294	3.332	3.346	3.377	3.408	3.435	3.493	2.586	2.629	2.633	2.642	2.648	2.684	2.697
3.501	3.537	3.554	3.562	3.628	3.852	3.871	2.726	2.770	2.773	2.800	2.809	2.818	2.821
3.886	3.971	4.024	4.027	4.225	4.395	5.020	2.848	2.880	2.954	3.012	3.067	3.084	3.090
							3.096	3.128	3.233	3.433	3.585	3.585	

Table 1: Sample Median, Scale Parameter, Shape Parameter, K-S and p value of the fitted generalized logistic models to data sets.

Data Set	Sample Median	Scale Parameter	Shape Parameter	K-S	p value
1	2.996	0.3643	1.1240	0.1135	0.3914
2	2.478	0.2745	0.9489	0.0492	0.9962

In this section the estimation of R when σ is known, is considered. Without loss of generality, we assume that $\sigma = 1$.

Table 2: Observed Frequencies, and Expected Frequencies for modified data set I when fitting the generalized logistic model.

Intervals	Observed Frequencies	Expected Frequencies	Chi-Square
< 2.5	12	10.5524	0.7401
2.5-3	20	18.5317	
3-3.5	17	19.9094	
3.5-4	9	9.7914	
> 4	5	4.2151	

Table 3: Observed Frequencies, and Expected Frequencies for modified data set II when fitting the generalized logistic model.

Intervals	Observed Frequencies	Expected Frequencies	Chi-Square
< 1.76	5	2.3900	0.6452
1.76-2.22	15	15.2904	
2.22-2.68	27	26.9100	
2.68-3.14	18	16.0011	
> 3.14	4	5.4027	

Table 4: MLE, Asymptotic Confidence Interval and Bootstrap Confidence Interval of R.

Data Set	Scale Parameter	Shape Parameter	$\hat{R}$	CI_{MLE}	CI_{Boot-p}	CI_{Boot-t}
1	0.319	1.0404	0.4962	(0.411,0.582)	(0.409,0.582)	(0.412,0.580)
2	0.319	1.0562				

3.1 MLE of R

Suppose $X_1, X_2, ..., X_n$ be a random sample from $PRHRM(\alpha, 1)$ and $Y_1, Y_2, ..., Y_m$ be a random from $PRHRM(\beta, 1)$. Based on Section 2, it is clear that the MLE of R will be

$$\hat{R} = \frac{m \sum_{i=1}^{n} \ln F(X_i)}{n \sum_{i=1}^{m} \ln F(Y_i) + m \sum_{i=1}^{n} \ln F(X_i)}. \tag{7}$$

It is easy to show that the $100(1-\gamma)\%$ confidence interval of R can be get as

$$\left[\frac{1}{1 + (\frac{1}{\hat{R}} - 1) F_{2n,2m;1-\gamma/2}}, \frac{1}{1 + (\frac{1}{\hat{R}} - 1) F_{2n,2m;\gamma/2}} \right] \tag{8}$$

where $F_{2n,2m;\gamma/2}$ and $F_{2n,2m;1-\gamma/2}$ are the lower and upper $\frac{\gamma}{2}$th percentile points of a F distribution with $2n$ and $2m$ degrees of freedom.

3.2 $UMVUE$ of R

Applying the results of Tong [3, 4], the $UMVUE$ of R is resulted asefore applying the results of Tong [3, 4], the $\tilde{R}$ is given by

$$\tilde{R} = \begin{cases} 1 - \sum_{i=0}^{m-1} (-1)^i \frac{(n-1)!(m-1)!}{(m-i-1)!(n+i-1)!} (\frac{T_1}{T_2})^i & \text{if} \quad T_1 \leq T_2 \\ \sum_{i=0}^{n-1} (-1)^i \frac{(n-1)!(m-1)!}{(m+i-1)!(n-i-1)!} (\frac{T_2}{T_1})^i & \text{if} \quad T_2 \leq T_1 \end{cases} \tag{9}$$

where $T_1 = -\sum_{i=1}^{n} \ln F(X_i)$ and $T_2 = -\sum_{i=1}^{m} \ln F(Y_i)$.

3.3 Bayes estimator of R

In this subsection, we attempt to find the Bayes estimator of R under the assumption that the shape parameters α and β are random variables. It is assume that α and β have independent gamma priors with the parameters $\alpha \sim Gamma(a_1, b_1)$ and $\beta \sim Gamma(a_2, b_2)$. Therefore,

$$\pi(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1 - 1} e^{-b_1 \alpha}; \qquad \alpha > 0 \tag{10}$$

and

$$\pi(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)} \beta^{a_2 - 1} e^{-b_2 \beta}; \qquad \beta > 0. \tag{11}$$

Here $a_1, b_1, a_2, b_2 > 0$. The posterior PDF's of α and β are as follows:

$$\alpha | data \sim Gamma(a_1 + n, b_1 + T_1), \tag{12}$$

$$\beta | data \sim Gamma(a_2 + m, b_2 + T_2), \tag{13}$$

where $T_1 = -\sum_{i=1}^{n} \ln F(X_i)$ and $T_2 = -\sum_{i=1}^{m} \ln F(Y_i)$. The Bayes estimate of R under squared error loss function can not be obtained analytically. Alternatively, via the approximation of Lindley [2], it can be seen that the approximate Bayes estimate of R, say $\hat{R}_{Bayes}$ under the squared error loss function is

$$\tilde{R} = \frac{\tilde{\beta}}{\tilde{\alpha} + \tilde{\beta}}, \ \tilde{\alpha} = \frac{n + a_1 - 1}{b_1 + T_1} \text{ and } \tilde{\beta} = \frac{m + a_2 - 1}{b_2 + T_2}.$$

4 Date Analysis

In this section, a data analysis of the strength data reported by Badar & Priest [1] is presented which are presented in the Table 3.
If we choose $F(x) = \frac{1}{1+e^{-x}}$ in (1) then the generalized logistic distribution is obtained as follows:

$$F(x;\mu,\sigma,\alpha) = \frac{1}{(1+e^{-\frac{x-\mu}{\sigma}})^{\alpha}}, \quad -\infty < x < +\infty \tag{14}$$

The generalized logistic distribution models to the two data sets, are fitted separately. The estimated scale and shape parameters are proposed assuming the location parameter to be known as the sample median for both the data sets. We also obtain Kolmogrov-Smirnov (K-S) distance between the empirical distribution functions, and the fitted distributions, and corresponding p values. All the results have reported in Table 1. For comparison purposes, we also compute the observed and the expected frequencies, the corresponding chi-square values based on the fitted models in the Tables 2 and 3. It is clear that generalized logistic model fits quite well to both the data sets. Because the two scale parameters are not very different, assuming the two parameters are equal, we are estimated the parameters and extracted the 95% confidence interval based on the asymptotic distribution of R and 95% percentile bootstrap method and bootstrap-t method confidence intervals. We reported the results in Table 4.

References

[1] Badar, M.G. and Priest, A.M. (1982), Statistical aspects fiber an bundle strength in hybrid composites, *Progress in Science and Engineering Composites*, 1129-1136.

[2] Lindley, D.V., 1980, Approximate Bayes method, *Trabajos de Estadistica*, *3*, 281-288.

[3] Tong, H. (1974), A note on the estimation of $P(Y < X)$ in the exponential case, *Technometrics*, *16* , 625.

[4] Tong, H. (1977), On the estimation of $P(Y < X)$ for exponential families, *IEEE Transactions on Reliability*, *26*, 54-56.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Likelihood ratio order of maxima from heterogeneous gamma random variables

Bashkar, E. [1] and Torabi, H. [2]

[1,2] Department of Statistics, Yazd University, Yazd, Iran

Abstract

In this paper, the largest order statistics arising from independent heterogeneous gamma random variables with respect to the likelihood ratio order are compared. Let $X_1, \ldots, X_n$
$(X_1^*, \ldots, X_n^*)$ be independent random variables where X_i (X_i^*) follows the gamma distribution with shape parameter α and scale parameter $\lambda_i(\lambda_i^*)$, in which $\alpha > 0$, $\lambda_i > 0$ $(\lambda_i^* > 0)$, $i = 1, \ldots, n$. Denote by $X_{n:n}$ and $X_{n:n}^*$ the corresponding largest order statistics, respectively. It is shown that, $X_{n:n}$ is stochastically larger than $X_{n:n}^*$ in terms of the likelihood ratio order if $\max\{\lambda_1, \ldots, \lambda_n\} \leq \min\{\lambda_1^*, \ldots, \lambda_n^*\}$. The result derived here strengthens and generalizes some known results in the literature.

Keywords: Gamma distribution, Likelihood ratio order, Parallel system, Order statistics.

1 Introduction

Order statistics play important rules in statistical inference, reliability theory, life testing, operations research and other related areas. Let $X_{1:n} \leq \ldots \leq X_{n:n}$ denote the order statistics arising from random variables $X_1, \ldots, X_n$. In reliability theory, the kth order statistic coresponds to the lifetime of a $(n-k+1)$-out-of-n system. In particular, $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. Order statistics have been extensively investigated in the case when the observations are independent and identically distributed (i.i.d.). In some practical situations, like in reliability theory,

[1] e.bashkar@stu.yazd.ac.ir
[2] htorabi@yazduni.ac.ir

observations are not necessarily i.i.d. Because of the complicated nature of the problem in the non-i.i.d. case, only limited works are found in the literature, for example, see [8, 3, 4].

However, [17] first compared order statistics from heterogeneous exponential samples, many researchers followed them to consider order statistics from various samples, including [18, 5, 13], [9], [10], [11, 12], [7, 14, 20, 28, 29], [22, 23, 24, 25, 26], [16, 1, 2, 27].

The gamma distribution is a flexible family of distributions commonly used in reliability and life testing areas. This distribution is a member of the family of distributions with decreasing, constant, and increasing failure rates in the cases $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$, respectively. It is known that the standard form probability density function of gamma distribution is

$$f(x;\alpha,\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0, \alpha > 0, \lambda > 0,$$

where α and λ are the shape parameter and scale parameter, respectively.

In this article, we study the maximum order statistics arising from independent heterogeneous gamma observations.

Throughout this paper, the terms increasing and decreasing are used for monotone non-decreasing and monotone non-increasing, respectively.

Let X and Y be two univariate random variables with distribution functions F and G, density functions f and g, and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. Random variable X is said to be smaller than Y in the

(i) likelihood ratio order, denoted by $X \leq_{lr} Y$, if $g(x)/f(x)$ is increasing in x;

(ii) hazard rate order, denoted by $X \leq_{hr} Y$, if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x;

(iii) reversed hazard rate order, denoted by $X \leq_{rh} Y$, if $\frac{G(x)}{F(x)}$ is increasing in x;

(iv) stochastic order, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for x.

For a comprehensive discussion on various stochastic orders, one can see [19].

It is well known that the notion of majorization is extremely useful and powerful in establishing various inequalities. For preliminary notations and terminologies on majorization theory, we refer the reader to [15]. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two real vectors and $x_{(1)} \leq \ldots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x}$.

Definition 1. The vector $\mathbf{x}$ is

(i) weakly supermajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\text{w}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \ldots, n$,

(ii) majorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\text{m}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$, for $j = 1, \ldots, n-1$.

Clearly, $\mathbf{x} \overset{\text{m}}{\preceq} \mathbf{y}$ implies $\mathbf{x} \overset{\text{w}}{\preceq} \mathbf{y}$.

Another interesting weaker order regarding to the majorization order introduced by [6] is the p-larger order which is defined as follows:

Definition 2. A vector $\mathbf{x}$ in $\mathbb{R}_+^n$ is said to be p-larger than another vector $\mathbf{y}$ in $\mathbb{R}_+^n$ (denoted by $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \quad j = 1, \ldots, n,$$

where $\mathbb{R}_+^n = \{(x_1, ..., x_n) | x_i \geq 0,\ i = 1, ..., n\}$.

It is known that for any two non-negative vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x} \overset{\text{m}}{\preceq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{p}}{\preceq} \mathbf{y}.$$

Let $X_1, \ldots, X_n$ $(X_1^*, \ldots, X_n^*)$ be independent random variables where X_i (Y_i) follows the gamma distribution with shape parameter α and scale parameter λ_i (λ_i^*), $i = 1, \ldots, n$. [20] showed that

$$(\lambda_1^*, \ldots, \lambda_n^*) \overset{\text{m}}{\preceq} (\lambda_1, \ldots, \lambda_n) \Longrightarrow X_{n:n}^* \leq_{st} X_{n:n}. \tag{1}$$

[10] relaxed the condition by proving that under the p-larger order,

$$(\lambda_1^*, \ldots, \lambda_n^*) \overset{\text{p}}{\preceq} (\lambda_1, \ldots, \lambda_n) \Longrightarrow X_{n:n}^* \leq_{st} X_{n:n}. \tag{2}$$

Afterward, [16] proved that

$$(\lambda_1^*, \ldots, \lambda_n^*) \overset{\text{w}}{\preceq} (\lambda_1, \ldots, \lambda_n) \Longrightarrow X_{n:n}^* \leq_{rh} X_{n:n}. \tag{3}$$

Further suppose $Y_1, \ldots, Y_n$ $(Z_1, \ldots, Z_n)$ be independent gamma random variables with shape parameter α and scale parameter $\bar{\lambda} = \frac{\sum_{k=1}^n \lambda_k}{n}$ $(\tilde{\lambda} = (\prod_{k=1}^n \lambda_k)^{\frac{1}{n}})$. If $0 < \alpha \leq 1$, [1] established that

$$Z_{n:n} \leq_{hr} X_{n:n}.$$

Lately, [27] showed that if $0 < \alpha \leq 1$, then

$$Y_{n:n} \leq_{lr} X_{n:n}. \tag{4}$$

They also posed a question whether the result in (4) can be established for the case when the shape parameter is larger than 1. In the current paper, we pose a generalization of this question and answer it; More precisely, we establish that for any $\alpha > 0$, $\max\{\lambda_1, \ldots, \lambda_n\} \leq \min\{\lambda_1^*, \ldots, \lambda_n^*\}$ implies $X_{n:n} \geq_{lr} X_{n:n}^*$. Note the result extends and sustains the corresponding result for the exponential case which is established in the literature.

2 Main result

First, we present the following lemma in order to prove our main result. [[21]] For $\alpha > 0$ and $y \in \mathbb{R}_+$, in which $\mathbb{R}_+ = [0, +\infty)$, the function

$$f(x) = x + \frac{y^{\alpha-1} e^{-xy}}{\int_0^y u^{\alpha-1} e^{-xu} du}$$

is increasing in $x \in \mathbb{R}_+$.

Now, we state our main result. For comparing two parallel systems with independent gamma components, we have the following result.

Theorem 1. *Let $X_1,\ldots,X_n$ $(X_1^*,\ldots,X_n^*)$ be independent random variables where X_i (X_i^*) follows the gamma distribution with shape parameter α and scale parameter λ_i (λ_i^*), $\alpha>0$, $\lambda_i>0$ $(\lambda_i^*>0)$, $i=1,\ldots,n$. Then, $\max\{\lambda_1,\ldots,\lambda_n\}\leq\min\{\lambda_1^*,\ldots,\lambda_n^*\}$ implies $X_{n:n}\geq_{lr}X_{n:n}^*$.*

Proof. Let $F_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})$ denote the distribution function of $X_{n:n}$ with corresponding density $f_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})$. We need to prove that the ratio of density functions

$$\frac{f_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})}{f_{X_{n:n}^*}(x;\alpha,\boldsymbol{\lambda^*})}=\frac{\sum_{i=1}^n\frac{e^{-\lambda_i x}}{\int_0^x u^{\alpha-1}e^{-\lambda_i u}du}}{\sum_{i=1}^n\frac{e^{-\lambda_i^* x}}{\int_0^x u^{\alpha-1}e^{-\lambda_i^* u}du}}\frac{F_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})}{F_{X_{n:n}^*}(x;\alpha,\boldsymbol{\lambda^*})}=:h(x)\frac{F_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})}{F_{X_{n:n}^*}(x;\alpha,\boldsymbol{\lambda^*})}$$

is increasing in $x>0$. It follows from (1) that $\frac{F_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})}{F_{X_{n:n}^*}(x;\alpha,\boldsymbol{\lambda^*})}$ is increasing in $x>0$; So, it suffices to show that $h(x)$ is increasing in x. Let $\varphi(x;\alpha,\boldsymbol{\lambda})=\sum_{i=1}^n\frac{e^{-\lambda_i x}}{\int_0^x u^{\alpha-1}e^{-\lambda_i u}du}=:\sum_{i=1}^n\psi(x,\alpha,\lambda_i)$. Thus $h(x)=\frac{\varphi(x;\alpha,\boldsymbol{\lambda})}{\varphi(x;\alpha,\boldsymbol{\lambda^*})}$. Taking derivative with respect to x, we get

$$h'(x)=\frac{\varphi'(x;\alpha,\boldsymbol{\lambda})\varphi(x;\alpha,\boldsymbol{\lambda^*})-\varphi'(x;\alpha,\boldsymbol{\lambda^*})\varphi(x;\alpha,\boldsymbol{\lambda})}{[\varphi(x;\alpha,\boldsymbol{\lambda^*})]^2}\overset{\text{sgn}}{=}$$

$$\left(\sum_{i=1}^n\frac{-\lambda_i e^{-\lambda_i x}\int_0^x u^{\alpha-1}e^{-\lambda_i u}du-x^{\alpha-1}e^{-\lambda_i x}e^{-\lambda_i x}}{[\int_0^x u^{\alpha-1}e^{-\lambda_i u}du]^2}\right)\varphi(x;\alpha,\boldsymbol{\lambda^*})$$

$$-\left(\sum_{i=1}^n\frac{-\lambda_i^* e^{-\lambda_i^* x}\int_0^x u^{\alpha-1}e^{-\lambda_i^* u}du-x^{\alpha-1}e^{-\lambda_i^* x}e^{-\lambda_i^* x}}{[\int_0^x u^{\alpha-1}e^{-\lambda_i^* u}du]^2}\right)\varphi(x;\alpha,\boldsymbol{\lambda})=$$

$$\sum_{i=1}^n\sum_{j=1}^n\psi(x,\alpha,\lambda_i^*)\psi(x,\alpha,\lambda_j)\left[(\lambda_i^*+\frac{x^{\alpha-1}e^{-\lambda_i^* x}}{\int_0^x u^{\alpha-1}e^{-\lambda_i^* u}du})-(\lambda_j+\frac{x^{\alpha-1}e^{-\lambda_j x}}{\int_0^x u^{\alpha-1}e^{-\lambda_j u}du})\right].$$

According to Lemma 2, $\max\{\lambda_1,\ldots,\lambda_n\}\leq\min\{\lambda_1^*,\ldots,\lambda_n^*\}$ implies $(\lambda_i^*+\frac{x^{\alpha-1}e^{-\lambda_i^* x}}{\int_0^x u^{\alpha-1}e^{-\lambda_i^* u}du})\geq(\lambda_j+\frac{x^{\alpha-1}e^{-\lambda_j x}}{\int_0^x u^{\alpha-1}e^{-\lambda_j u}du})$. Thus $\frac{f_{X_{n:n}}(x;\alpha,\boldsymbol{\lambda})}{f_{X_{n:n}^*}(x;\alpha,\boldsymbol{\lambda^*})}$ increases in $x>0$. That is, $X_{n:n}\geq_{lr}X_{n:n}^*$. □

Note that it is swimmingly seen that the result in Theorem 1, strengthens and generalizes the results in [9], [11] and [14] for the case of exponential distributions. The above theorem immediately leads to the following corollary. Let $X_1,\ldots,X_n$ be independent gamma random variables where X_i follows the gamma distribution with shape parameter α and scale parameter λ_i, $i=1,\ldots,n$, and $Y_1,\ldots,Y_n$ be a random sample of size n from a gamma distribution with shape parameter α and common scale parameter $\lambda\geq\max\{\lambda_1,\ldots,\lambda_n\}(\leq\min\{\lambda_1,\ldots,\lambda_n\})$. Then, $X_{n:n}\geq_{lr}(\leq_{lr})Y_{n:n}$.

3 Conclusions

In this paper, we answered a generalization of an open problem regarding to the largest order statistics arising from independent heterogeneous gamma distributions with respect to likelihood ratio order, posed by [27].

References

[1] Balakrishnan, N. and Zhao, P. (2013a), Hazard rate comparison of parallel systems with heterogeneous gamma components, Journal of Multivariate Analysis, 113, 153-160.

[2] Balakrishnan, N. and Zhao, P. (2013b), Ordering properties of order statistics from heterogeneous populations: A review with an emphasis on some recent developments, Probability in the Engineering and Informational Sciences, 27, 403-469.

[3] Balakrishnan, N. and Rao, C.R., (Eds.) (1998a), Handbook of Statistics, in: Order Statistics: Theory and Methods. vol. 16, Elsevier, Amsterdam.

[4] Balakrishnan, N. and Rao, C.R., (Eds.) (1998b), Handbook of Statistics, in: Order Statistics: Applications. vol. 17, Elsevier, Amsterdam.

[5] Boland P.J., EL-Neweihi E. and Proschan, F. (1994), Schur properties of convolutions of exponential and geometric random variables, Journal of Multivariate Analysis, 48, 157-167.

[6] Bon, J.L. and Păltănea, E. (1999), Ordering properties of convolutions of exponential random variables, Lifetime Data Analysis, 5, 185-192.

[7] Bon, J.L. and Păltănea, E. (2006), Comparisons of order statistics in a random sequence to the same statistics with i.i.d. variables, ESAIM: Probability and Statistics, 10, 1-10.

[8] David, H.A. and Nagaraja. H.N. (2003), Order Statistics. 3rd ed., Wiley, Hoboken, New Jersey.

[9] Dykstra, R., Kochar, S.C. and Rojo, J. (1997), Stochastic comparisons of parallel systems of heterogeneous exponential components, Journal of Statistical Planning and Inference, 65, 203-211.

[10] Khaledi, B.-E., Farsinezhad, S. and Kochar, S.C. (2011), Stochastic comparisons of order statistics in the scale model, Journal of Statistical Planning and Inference, 141, 276-286.

[11] Khaledi, B.-E. and Kochar, S.C. (2000), Some new results on stochastic comparisons of parallel systems, Journal of Applied Probability, 37, 283-291.

[12] Khaledi, B.-E. and Kochar, S.C. (2007), Stochastic orderings of order statistics of independent random variables with different scale parameters, Communications in Statistics-Theory and Methods, 36, 1441-1449.

[13] Kochar, S.C. and Rojo, J. (1996), Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions, Journal of Multivariate Analysis, 59, 272-281.

[14] Kochar, S.C. and Xu, M. (2007), Stochastic comparisons of parallel systems when components have proportional hazard rates, Probability in the Engineering and Informational Sciences, 21, 597-609.

[15] Marshall A.W., Olkin, I. and Arnold, B.C. (2011), Inequalities: Theory of Majorization and its Applications. second ed., Springer, New York .

[16] Misra, N. and Misra, A.K. (2013), On comparison of reversed hazard rates of two parallel systems comprising of independent gamma components, Statistics and Probability Letters, 83, 1567-1570.

[17] Pledger, P. and Proschan, F. (1971), Comparison of order statistics and of spacings from heterogeneous distributions, In: Rustagi, J.S. (Ed.), Optimizing Methods in Statistics. Academic Press, NewYork, pp. 89-113.

[18] Proschan, F. and Sethuraman, J. (1976), Stochastic comparisons of order statistics from heterogeneous populations with applications in reliability, Journal of Multivariate Analysis, 6, 608-616.

[19] Shaked, M. and Shanthikumar, J.G. (2007), Stochastic Orders, Springer, New York.

[20] Sun, L. and Zhang, X. (2005), Stochastic comparisons of order statistics from gamma distributions, Journal of Multivariate Analysis, 93, 112-121.

[21] Zhao, P. (2011), On parallel systems with heterogeneous gamma components, Probability in the Engineering and Informational Sciences, 25, 369-391.

[22] Zhao, P. and Balakrishnan, N. (2009), Characterization of MRL order of fail-safe systems with heterogeneous exponentially distributed components, Journal of Statistical Planning and Inference, 139, 3027-3037.

[23] Zhao, P. and Balakrishnan, N. (2011a), MRL ordering of parallel systems with two heterogeneous components, Journal of Statistical Planning and Inference, 141, 631-638.

[24] Zhao, P. and Balakrishnan, N. (2011b), Some characterization results for parallel systems with two heterogeneous exponential components, Statistics, 45, 593-604.

[25] Zhao, P. and Balakrishnan, N. (2011c), Dispersive ordering of fail-safe systems with heterogeneous exponential components, Metrika, 74, 203-210.

[26] Zhao, P. and Balakrishnan, N. (2013), Comparisons of Largest Order Statistics from Multiple-outlier Gamma Models, Methodology and Computing in Applied Probability, 1-29.

[27] Zhao, P. and Balakrishnan, N. (2014), A stochastic inequality for the largest order statistics from heterogeneous gamma variables, Journal of Multivariate Analysis, 129, 145-150.

[28] Zhao, P., Li, X. and Balakrishnan, N. (2008), Conditional ordering of k-out-of-n systems with independent but non-identical components, Journal of Applied Probability, 45, 1113-1125.

[29] Zhao, P., Li, X. and Balakrishnan, N. (2009), Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables, Journal of Multivariate Analysis, 100, 952-962.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

A concept of extended cumulative residual entropy and its relation to reliability measures

Bazari, N. [1]

[1] Department of Statistics, University of Isfahan

Abstract

In this paper we propose a notion of cumulative residual entropy (ECRE) of a lifetime random variable extending the concept which was introduced by Psarrakos and Navarro (2012). The proposed criterion is closely related to k-record values. We also consider a dynamic version of ECRE based on residual lifetime and study its relation to reliability measures.

Keywords:

extended cumulative residual entropy , failure rate, k-record values

1 Introduction

Shannon entropy, which was first proposed by Claude Shannon is an appropriate criterion for measuring uncertainty of a random variable. For a continuous distribution F with density f, the Shannon entropy, called also differential entropy, is given by

$$H(F) = -\int f(x)\log f(x)dx \tag{1}$$

Rao et al.[4] proposed an alternative measure of uncertainty, called cumulative residual entropy(CRE), by replacing the probability density function with the reliability function $\bar{F} = 1 - F$. This measure is as follows

$$\xi(X) = -\int_0^\infty \overline{F}(t)\log \overline{F}(t)dt \tag{2}$$

[1] n.bazari@sci.ui.ac.ir

Several properties of the CRE were established in Asadi and Zohrevand [1]. Recently Psarrakos and Navarro [3] extended the concept of CRE by using mean time between record values, and defined the generalized cumulative residual entropy (GCRE) of a continuous random variable X as

$$\xi_n(X) = \int_0^\infty \overline{F}(x)\frac{[\Lambda(x)]^n}{n!}dx \tag{3}$$

for n=1,2,..., where $\Lambda(x) = -\log \overline{F}(x)$ is the cumulative hazard function.
In this paper, what Psarrakos and Navarro [3] had proposed in their work, will be examined from another perspective. In fact, we develop the concept of cumulative residual entropy by adding a new parameter to the measure of Psarrakos and Navarro (2012). It is shown that the proposed measure is related to the difference between mean of k-record values.

2 Main Results

Before giving the main results of the paper, we give the definition of kth records.

Definition 1. *Let $X_i, i \geq 1$ be an iid sequence of r.v.'s with a continuous distribution function F and probability density function f. For $k \geq 1$,the r.v.'s $L^{(k)}(n)$ given by*
$L^{(k)}(1) = 1$,
$L^{(k)}(n+1) = min\{j \geq 1; X_{j:j+k-1} > X_{L^{(k)}(n):L^{(K)}(n)+k-1}\}, n = 1, 2, ...$
are called k-th record times and the quantities $X_{L^{(k)}(n):L^{(K)}(n)+k-1}$ which are denoted by $X_{L^{(k)}(n)}, n = 1, 2, ...$, are termed k-th record values or k-records.

The reliability function of k-record, $\overline{F}_{k,n}$, $k \geq 1$, is

$$\overline{F}_{k,n}(x) = [\overline{F}(x)]^k \sum_{j=0}^{n} \frac{1}{j!}[k\Lambda(x)]^j \tag{4}$$

and the probability density function of k-record, $f_{k,n}$ is

$$f_{k,n}(x) = \frac{k^n}{(n-1)!}[\Lambda(x)]^{n-1}[\overline{F}(x)]^{k-1}f(x) \tag{5}$$

Now we consider the mean value of $F_{k,n}$, $\mu_{k,n} = \int_0^\infty \overline{F}_{k,n}(x)dx, n, k \geq 1$

$$\mu_{k,n+1} - \mu_{k,n} = \int_0^\infty [\overline{F}(x)]^k \frac{[k\Lambda(x)]^n}{n!} \tag{6}$$

Letting $k = 1$ in (6) yields (3), and $k = 1, n = 1$ we have (2). These facts inspire us to define the extended cumulative residual entropy (ECRE) of X as

$$\xi_{k,n} = \int_0^\infty [\overline{F}(x)]^k \frac{[k\Lambda(x)]^n}{n!}$$

Notice from (6), the ECRE can be written as

$$\xi_{k,n} = \int_0^\infty [\overline{F}(x)]^{(k-1)} f(x)\frac{\overline{F}(x)}{f(x)}\frac{[k\Lambda(x)]^n}{n!} = \frac{1}{k}E(\frac{1}{\lambda(X_{k,n+1})}) \tag{7}$$

for $n \geq 0, k \geq 1$, where $\lambda = f/\overline{F}$ is the failure rate function of F and $X_{k,n+1}$ is a random variable with reliability $\overline{F}_{k,n+1}$.
Using relationship between CRE and expectation of mean residual lifetime, which proved in Asadi and Zohrevand [1], we can show following theorem.

Theorem 1. *If $\xi_{k,n}$ is an ECRE based on the sequence of k-records then, for n=1, $\xi_{k,1}$ is decreasing function in k.*

Let $X_{k,n}$ be a sequence of k-records, then

(i) $X_{k,n} \leq_{lr} X_{k,n+1}$

(ii) $X_{k,n} \geq_{lr} X_{k+1,n}$

where $\leq_{lr}$ denotes the likelihood ratio order. This lemma implies following theorem.

Theorem 2. *If X is IFR(DFR), then*

(i) $\xi_{k,n} \geq (\leq) \xi_{k,n+1}$

(ii) $\xi_{k,n} \leq (\geq) \xi_{k+1,n}$

In the following theorem we show that under what condition ,the ECREs can be compared.

Theorem 3. *If $X \leq_{hr} Y$ and either X or Y are DFR, then*

$$\xi_{k,n}(X) \leq \xi_{k,n}(Y)$$

where $\leq_{hr}$ denotes the hazard rate order.
Similarly we can also consider a dynamic version of DCRE, that is, the ECRE of the residual lifetime $X_t = (X - t|X > t)$. This is given by

$$\xi_{k,n}(X;t) = \xi_{k,n}(x_t) = \frac{1}{n!} \int_t^\infty [\frac{\overline{F}(x)}{\overline{F}(t)}]^k [-k \log \frac{\overline{F}(x)}{\overline{F}(t)}]^n dx \tag{8}$$

This function is called dynamic extended cumulative residual entropy (DECRE).
We can study monotony properties of DECRE by using following lemma. If X is absolutely continuous, then

$$\xi_{k,n}^{'}(X;t) = k\lambda(t)[\xi_{k,n}(X;t) - \xi_{k,n-1}(X;t)]$$

for $n \geq 1, k \geq 1$.

Theorem 4. *If X is IFR (DFR), then $\xi_{k,n}(X;t)$ is decreasing (increasing) for $n \geq 0$ and $k \geq 1$*

References

[1] Asadi, M. and Zohrevand, Y. (2007), On the dynamic cumulative residual entropy, *Journal of Statistical Planning and Inference*, **137**, 1931-1941.

[2] Kamps, U. (1995), *A Concept of Generalized Order Statistics*, Teubner, Stuttgart.

[3] Psarrakos, G. and Navarro, J. (2012), Generalized cumulative residual entropy. *Metrika*, **76**, 623-640.

[4] Rao, M. Chen, Y. Vemuri, BC. and vang, F. (2004),Cumulative residual entropy: a new measure of information. *IEEE Trans Inf Theory*, **50**, 1220-1228.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Measure of uncertainty in the residual lifetime of a coherent system

Chahkandi, M. [1] and Toomaj, A. [2]

[1] Department of Statistics, University of Birjand, Birjand, Iran
[2] Department of Statistics, Gonbad Kavous University, Gonbad Kavous, Iran

Abstract

In this paper, we provide an expression for the entropy of the residual lifetime of a coherent system, especially an i-out-of-n system, when all components of the system are alive at time t. Various properties of the proposed entropy are discussed. Some partial ordering results among the system residual lifetimes in terms of their entropies are also obtained.

Keywords: Entropy , Residual lifetime, Signature

1 Introduction

Consider an absolutely continuous non-negative random variable X with cumulative distribution function $F(x)$ and density function $f(x)$. Shannon entropy is defined as

$$H(f) = H(X) = E(-\log f(X)) = -\int_0^\infty f(x)\log f(x)dx,$$

where "log" means natural logarithm (see e.g. [3]). The measure $H(f)$ gives expected uncertainty contained in $f(x)$ about the predictability of an outcome of X. Several properties, generalizations and applications of the mentioned measure can be found in [1]-[7], [9], [10] and the references therein. The Shannon entropy is a useful criterion for measuring the uncertainty of the system lifetime, but it is no longer useful for measuring the

[1]mchahkandi@birjand.ac.ir
[2]ab.toomaj@gonbad.ac.ir

uncertainty of a used system. A more realistic measure which makes use of the age in calculating the expected uncertainty contained in $f(x)$, is given in [4]. Given that the unit has survived up to time t $(X > t)$, the entropy of the remaining lifetime of X, is given by

$$\begin{aligned} H(f;t) &= -\int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx \\ &= 1 - \frac{1}{\bar{F}(t)} \int_t^\infty \log(\lambda(x)) f(x) dx, \end{aligned} \tag{1}$$

where $\lambda(x)$ is the failure rate function of X. When the component lifetime has survived up to time t, $H(f;t)$ measures the expected uncertainty contained in the conditional density of $X - t$ given $X > t$ about the predictability of remaining lifetime of the unit.

Let T denote the lifetime of a coherent system consisting of n independent and identically distributed (i.i.d.) component lifetimes $X_1, \ldots, X_n$. In the present paper, under the condition that at time t all components of the system are alive $(X_{1:n} > t)$, we are interested in measuring the expected uncertainty contained in the conditional density of $T - t$ about the predictability of remaining lifetime of the system. For this purpose, at first, we find an expression for the proposed entropy of an i-out-of-n system and then extend the results to a coherent system by using the notion of system signature.

2 Main result

Let $T_t^{1,i,n} = X_{i:n} - t | X_{1:n} > t,\ i = 1, 2, \cdots, n$, denote the residual lifetime of an i-out-of-n system consisting of n i.i.d. components with lifetimes $X_1, \cdots, X_n$ having the common cdf F in which, at time t, all components of the system are working. It follows that

$$\bar{F}_{T_t^{1,i,n}}(x) = \sum_{k=0}^{i-1} \binom{n}{k} \left(1 - \bar{F}_t(x)\right)^k \left(\bar{F}_t(x)\right)^{n-k},\ x \geq 0, \tag{2}$$

where $\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)},\ t > 0$. The probability integral transformation $W_i = \bar{F}_t(T_t^{1,i,n})$ has the beta distribution with parameters $n - i + 1$ and i and with the following density function

$$g_i(w) = \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} (1-w)^{i-1} w^{n-i}, \quad 0 < w < 1,\ i = 1, \cdots, n. \tag{3}$$

Ebrahimi *et al.* [6] showed that the expression for the beta entropy is

$$H(W_i) = \log B(i, n-i+1) - (n-i)[\psi(n-i+1) - \psi(n+1)] - (i-1)[\psi(i-1) - \psi(n+1)]. \tag{4}$$

In the forthcoming theorem, we provide an expression for $H(T_t^{1,i,n})$ by using the earlier mentioned transform.

Theorem 1. *Let $T_t^{1,i,n}$ be the residual lifetime of an i-out-of-n system in which, at time t, all components of the system are operating. It follows that*

$$\begin{aligned} H(T_t^{1,i,n}) &= H(W_i) - E[\log f_t(T_t^{1,i,n})] = H(W_i) - E[\log f_t(\bar{F}_t^{-1}(W_i))] \\ &= H(W_i) + \log \bar{F}(t) - E[\log f(\bar{F}_t^{-1}(W_i) + t)]. \end{aligned} \tag{5}$$

Proof. By noting $W_i = \bar{F}_t(T_t^{1,i,n})$ and using the entropy transformation formula in [5] the proof can be obtained. □

In the following theorem, we provide some bounds in terms of the entropy of the parent residual lifetime distribution. The proof can be found from the results of [6].

Theorem 2. *Consider* $T_t^{1,i,n} = X_{i:n} - t | X_{1:n} > t,\ i = 1, 2, \cdots, n$. *Then*

$$\begin{aligned} H(T_t^{1,i,n}) &\geq H(W_i) + g_i(p_i)[H(X_t) + I(A)], \\ H(T_t^{1,i,n}) &\leq H(W_i) + g_i(p_i)[H(X_t) + I(\bar{A})], \end{aligned}$$

where $p_i = \frac{n-i}{n-1}$ *is the mode of* (3), $X_t = X - t | X > t$, $A = \{x : f_t(x) \leq 1\}$, $\bar{A} = \{x : f_t(x) > 1\}$ *and* $I(A) = \int_A f_t(x) \log f_t(x) dx$.

Using the concept of dispersive order and Theorem 3.B.25 of [12], we have the next result. To see the other concepts of ageing notions and stochastic orders, we refer the reader to [12].

Theorem 3. *If* X *is IFR (DFR), then*

$$H(T_t^{1,i,n}) \leq (\geq) H(X_{i:n}),$$

for all $t \geq 0$, *where* $X_{i:n}$ *denotes the* i*th order statistic.*

Some properties of the proposed entropy are given in the next theorems. The next theorem can be found by using Lemma 2.1 of [2].

Theorem 4. *If* $H(f;t)$ *is decreasing in* t, *then* $H(T_t^{1,n,n})$ *is decreasing in* t.

Theorem 5. *Let* $X_1, \ldots, X_n$ *be a set of i.i.d. random variables from a DFR distribution* F.

a) $H(T_t^{X,1,i,n})$ *is increasing in* i.

b) $H(T_t^{X,1,1,n})$ *is decreasing in* n.

Proof. Using the results of [8] and [6], the proof can be found. □

In the next theorems, we compare the system residual lifetimes in terms of their entropies. Note that $\leq_{disp}$ and $\leq_{\ell r}$ denote the dispersive and likelihood ratio ordering, respectively.

Theorem 6. *Let* $T_t^{X,1,i,n}$ *and* $T_t^{Y,1,i,n}$ *denote the residual lifetimes of two systems consisting of* n *i.i.d. components lifetimes of cdfs* F *and* G, *respectively. If* $X \leq_{disp} Y$, *then* $H(T_t^{X,1,i,n}) \leq H(T_t^{Y,1,i,n})$, *for all* $t \geq 0$.

Theorem 7. *Let* $X_1, \ldots, X_n$ *be a random sample of size* n *from a continuous distribution* F *and* $Y_1, \ldots, Y_m$ *be a random sample of size* m *from another continuous distribution* G. *If either* F *or* G *is DFR and* $X \leq_{disp} Y$, *then* $H(T_t^{X,1,i,n}) \leq H(T_t^{Y,1,j,m})$ *for all* $t \geq 0$ *and* $i \leq j$, $n - i \geq m - j$.

Theorem 8. *Let* $X_1, \ldots, X_n$ *be a random sample of size* n *from a continuous distribution* F *and* $Y_1, \ldots, Y_m$ *be a random sample of size* m *from another continuous distribution* G, *where* $n \geq m$. *If either* F *or* G *is DFR and* $X \leq_{\ell r} Y$, *then* $H(T_t^{X,1,1,n}) \leq H(T_t^{Y,1,1,m})$.

By using the concept of system signature, we can extend the previous results to a coherent system with an arbitrary structure. For more details on the concept of system signature, see e.g. [4]. Let us consider a coherent system consisting of n i.i.d. component lifetimes $X_1, \ldots, X_n$, with signature vector $\mathbf{s} = (s_1, \ldots, s_n)$. Let $T_t^{1,n} = T - t | X_{1:n} > t$ denote the residual lifetime of the system by knowing that at time t all of the components are working. Then we have

$$\begin{aligned} H(T_t^{1,n}) &= H(V) - E(\log f_t(T_t^{1,n})) \\ &= H(V) - \sum_{i=1}^{n} s_i E(\log f_t(\bar{F}_t^{-1}(W_i))) \\ &= H(V) - \sum_{i=1}^{n} s_i H(W_i) + \sum_{i=1}^{n} s_i H(T_t^{1,i,n}), \end{aligned} \tag{6}$$

where $H(W_i)$ is given in (4) and $H(V)$ denotes the entropy of a system the density function $g_v(v) = \sum_{i=1}^{n} s_i g_i(v),\ 0 < v < 1$.

References

[1] Asadi, M. Ebrahimi, N. and Soofi, E. (2005). Dynamic generalized information measures, *Statist. Probab. Lett.* **71**, pp. 89-98.

[2] Asadi, M. and Ebrahimi, N. (2000). Residual entropy and its characterizations in terms of hazard function and mean residual life function. Statistics and Probability Letters, **49**, 263-269.

[3] Cover, T. A. and Thomas, J. A.(2006). *Elements of information theory*, New Jersey: Wiley and Sons, Inc.

[4] Ebrahimi, N. (1996). How to measure uncertainty in the residual lifetime distributions. *Sankhya A*, **58**, 48-57.

[5] Ebrahimi, N. Soofi, E.S. and Soyer, R. (2010). Information measures in perspective, *Int. Stat. Rev.* **78**, pp. 383-412.

[6] Ebrahimi, N. Soofi, E.S. and Zahedi, H. (2004). Information properties of order statistics and spacings, *IEEE T. Inform. Theory.* **46**, pp. 209-220.

[7] Ebrahimi, N. and Pellerey, F, (1995). New Partial Ordering of Survival Functions Based on the Notion of Uncertainty. *Journal of Applied Probability.* **32**, pp. 202-211.

[8] Kochar, S. C. (1996). Dispersive ordering of order statistics, *Statistics and Probability Letters.* **27**, pp. 271-274.

[9] Rao, M., Chen, Y., Vemuri, B. C. and Wang, F. (2004). Cumulative residual entropy: A new measure of information, *IEEE Transactions on Information Theory.* **50**, 1220-1228.

[10] Toomaj, A. and Doostparast, M. (2014). A note on signature-based expressions for the entropy of mixed r-out-of-n systems, *Nav. Res. Log.* **61**, pp. 202-206.

[11] Samaniego, F. J. (2007). *System Signatures and their Applications in Engineering Reliability*. Springer: New York,.

[12] Shaked, M. and Shanthikumar, J.G. (2007). *Stochastic orders*. Springer Verlag, New York.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Dependence structure based guidelines for active redundancy allocation problems in engineering systems

Jeddi, H. [1] and Doostparast, M. [2]

[1,2] Department of Statistics, Ferdowsi University of Mashhad, Mashhad , Iran

Abstract

In this paper, we consider the problem of allocation a spare component to existing components in order to improve a system reliability. Here, we assumed that the component lifetimes are dependent and heterogeneous. Active redundancy scheme is investigated for 2-component systems. When there is the (positive/negative) quadratic dependence among the component lifetimes, the results are simplified.

Keywords: Copula, Quadratic Dependence, Redundancy, Reliability.

1 Introduction

Redundancy policies are usually used to increase the reliability of a given system. In common, there are two schemes to allocate redundant components to the system, called *active* and *standby* redundancy allocations. In the active redundancy policy, the redundant components are put in parallel to the components of the system while under the standby policy, the redundant components start functioning immediately after the failure of a component.

There are many researches for deriving optimal redundancy allocation polices (if exists). Among them, Boland, *et al.* [3] considered some stochastic orders for k-out-of-n systems, which the system work whenever at least k components work. Assuming that the components are independent and identically distributed (i.i.d.), they proved that under the active redundancy scheme for series systems, the optimal policy always allocates the

[1]je57@stu-mail.um.ac.ir
[2]doustparast@um.ac.ir

spare to the weakest component. For standby redundancy and under the *likelihood ratio ordering*, Boland *et al.* [3] gives also sufficient conditions in which for series systems, allocation should be given to the weakest component while in parallel systems, it should allocate the spare component to the strongest one. For recent results on the redundancy allocation problem (RAP) for system with independent components, see, e.g., Valdes and Zequeira [11], Romera *et al.* [9], Hu and Wang [4] and the references therein. In the above-mentioned works, it is usually assumed that the component lifetimes are independent. Recently, Belzunce *et al.* [1, 2], considered the RAP when component lifetimes are dependent. For modelling the dependency between the component lifetimes and comparison purpose, they used the concept of *joint stochastic orders.*
In this paper, we consider the RAP without any restriction to a special structure for dependency among component lifetimes and the conditions are given in terms of the joint distribution of the component lifetimes. Therefore, the rest of this paper is organized as follows: In Section 2, the RAP is considered for 2-component systems under active policy. Quadratic dependence orders among component lifetimes are used to derived the optimal policy. Also, various examples are given in Section 2.

2 Main results

In this section, we consider series systems with two dependent components. First, we provide a formal definition of (usual) statistical order.

Definition 1 (Shaked and Shantikumar, [10]). *Let $F_X(t) = P(X \leq t)$ and $F_Y(t) = P(Y \leq t)$, for $t > 0$, be distribution functions (DFs) of lifetime variables X and Y, respectively. Then X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if $\overline{F}_X(t) \leq \overline{F}_Y(t), \forall t > 0$, where $\overline{F}_X(t) = 1 - F_X(t)$ and $\overline{F}_Y(t) = 1 - F_Y(t)$ are the survival functions (SFs) of X and Y, respectively.*

Let $X_i (i = 1, 2)$ and S stand for the $i - th$ component lifetime and spare component lifetime, respectively. The component lifetimes X_1, X_2 and S are also dependent. This assumption is naturally consistent since components are working under the same loading and environment situations and etc; See, e.g. Nelsen [8].

2.1 RAP in series system

At first, we assume that series systems where the spare component is allocated under the active redundancy scheme. Here, "$\wedge\{a, b\}$" and "$\vee\{a, b\}$" stand for the minimum and the maximum of real numbers a and b, respectively.

Theorem 1. *Let $T_1 = \wedge\{\vee\{X_1, S\}, X_2\}$ and $T_2 = \wedge\{X_1, \vee\{X_2, S\}\}$. Then $T_1 \geq_{st} T_2$ if and only if*

$$P(X_1 > a, S > a) \leq P(X_2 > a, S > a), \forall a > 0. \tag{1}$$

Proof. By definition, $\overline{F}_{T_1}(a) = P\left(\vee(X_1, S) > a, X_2 > a\right)$ for all $a > 0$. Since $\{\vee(X_1, S) > a\} = \{X_1 > a\} \cup \{X_1 < a, S > a\}$, we conclude that

$$\overline{F}_{T_1}(a) = P(X_1 > a, X_2 > a) + P(X_1 < a, X_2 > a, S > a). \tag{2}$$

Similarly

$$\overline{F}_{T_2}(a) = P(X_1 > a, X_2 > a) + P(X_1 > a, X_2 < a, S > a).$$

So $T_1 \geq_{st} T_2$ if and only if, for all $a > 0$,

$$P(X_1 < a, X_2 > a, S > a) \geq P(X_1 > a, X_2 < a, S > a). \tag{3}$$

Notice that

$$P(X_1 < a, X_2 > a, S > a) = P(S > a, X_2 > a) - P(X_1 > a, X_2 > a, S > a),$$

and

$$P(X_1 > a, X_2 < a, S > a) = P(S > a, X_1 > a) - P(X_1 > a, X_2 > a, S > a).$$

Hence, $\overline{F}_{T_1}(a) \geq \overline{F}_{T_2}(a)$ if and only if $P(S > a, X_2 > a) \geq P(S > a, X_1 > a)$, for all $a > 0$ and the proof is complete. □

From Equation (1), one can easy verify that, for $i = 1, 2$,

$$\begin{aligned} P(X_i > a, S > a) &= 1 - P(X_i \leq a \cup S \leq a) \\ &= 1 - P(X_i \leq a) - P(S \leq a) + P(X_i \leq a, S \leq a) \\ &= 1 - F_{X_i}(a) - F_S(a) + F_{X_i,S}(a, a) \\ &= \overline{F}_{X_i}(a) - F_S(a) + F_{X_i,S}(a, a). \end{aligned}$$

Under the assumption of Theorem 1, $T_1 \geq_{st} T_2$ if and only if

$$F_{X_2}(a) + F_{X_1,S}(a, a) \leq F_{X_1}(a) + F_{X_2,S}(a, a), \forall a > 0. \tag{4}$$

Notice that in Theorem 1, the given condition in Equation (1) is defined on the basis of pairwise dependences in random vectors (X_1, S) and (X_2, S) while the dependence in (X_1, X_2) does not play any role.

Suppose that lifetime S is independent of (X_1, X_2). Then, $T_1 \geq_{st} T_2$ if and only if $X_1 \leq_{st} X_2$.

Proof. It immediately follows by Theorem 1. □

Theorem 1, Corollary 2.1 and Proposition 2.1 extend Lemma 2.2 of Boland *et al.* (1992) where they assumed that, the lifetimes X_1, X_2 and S are independent. If $(X_1|S = a) \leq_{st} (X_2|S = a)$ for all $a > 0$, then $T_1 \geq_{st} T_2$.

Proof. By definition for all $a > 0$, we have

$$\begin{aligned} P(X_1 > a, S > a) &= \int_a^\infty P(X_1 > a|S = s) dF_S(s) \leq \int_a^\infty P(X_2 > a|S = s) dF_S(s) \\ &= P(X_2 > a, S > a). \end{aligned} \tag{5}$$

Then by Theorem 1, the proof is complete. □

There are many engineering systems in which Condition (1) satisfies. A natural situation occurs when component lifetimes exhibit dependent behaviour due the same environment conditions such as loading, pressure and etc. In other words, since the components are working under same circumstances, it is usually observed that there exists some dependence behaviour among component lifetimes. Lehmann [6] introduced the notion of *positive (negative) quadrant dependent* which is useful for modelling dependent random variables.

Definition 2 (Lehmann, [6]). *The random variables X and Y are said positive quadrant dependent (PQD) if for every $y_1, y_2 \in \mathbb{R}$,*

$$F_{Y_1,Y_2}(y_1, y_2) \geq F_{Y_1}(y_1)F_{Y_2}(y_2), \tag{6}$$

and negative quadrant dependent (NQD) if for every $y_1, y_2 \in \mathbb{R}$,

$$F_{Y_1,Y_2}(y_1, y_2) \leq F_{Y_1}(y_1)F_{Y_2}(y_2). \tag{7}$$

If (X_1, S) and (X_2, S) be PQD and NQD, respectively, and $X_1 \leq_{st} X_2$ then $T_1 \geq_{st} T_2$.

Proof. Inequalities (6) and (7) are equivalent to $\bar{F}_{Y_1,Y_2}(y_1, y_2) \geq \bar{F}_{Y_1}(y_1)\bar{F}_{Y_2}(y_2)$ and $\bar{F}_{Y_1,Y_2}(y_1, y_2) \leq \bar{F}_{Y_1}(y_1)\bar{F}_{Y_2}(y_2)$, respectively, where $\overline{F}_{X,Y}(x, y) = P(X > x, Y > y)$ is the joint survival function of X and Y. The proof is immediately concluded from Theorem 1 and Definition 2. □

Lehmann's idea on PQD is used as a base for introducing some stochastic orders including *positive (negative) quadratic order* in the literature.

Definition 3 (Shaked and Shantikumar, [10]). *Let (U_1, U_2) and (V_1, V_2) be two random vectors with joint DFs $F_{U_1,U_2}(.,.)$ and $F_{V_1,V_2}(.,.)$, respectively. Furthermore, F_{U_1,U_2} and F_{V_1,V_2} have the same univariate marginals. If*

$$F_{U_1,U_2}(a, b) \leq F_{V_1,V_2}(a, b), \forall a, b, \tag{8}$$

then (U_1, U_2) is called smaller than (V_1, V_2) in the PQD order, denoted by $(U_1, U_2) \leq_{PQD} (V_1, V_2)$.

Now, we provide sufficient conditions for (1) in terms of quadratic dependence structure. If $X_1 \overset{D}{=} X_2$ and $(X_1, S) \leq_{PQD} (X_2, S)$ then Condition (1) holds and hence $T_1 \geq_{st} T_2$. Now, we provide some illustrative examples.

Example 1. *(FGM copula) Let (X_1, X_2, S) follows the tri-variate Farlie-Gumbel-Morgenstern (FGM) distribution with the joint DF*

$$\begin{aligned} F_{X_1,X_2,S}(x_1, x_2, s) &= F_{X_1}(x_1)F_{X_2}(x_2)F_S(s) \\ &\times \{1 + \theta_{12}\overline{F}_{X_1}(x_1)\overline{F}_{X_2}(x_2) + \theta_{13}\overline{F}_{X_1}(x_1)\overline{F}_S(s) + \theta_{23}\overline{F}_{X_2}(x_2)\overline{F}_S(s)\} \end{aligned} \tag{9}$$

for $x_1 > 0, x_2 > 0, s > 0$ and $-1 < \theta_{12} < 1, -1 < \theta_{13} < 1$, and $-1 < \theta_{23} < 1$. Notice that θ_{12}, θ_{13} and θ_{23} are the parameters of the FGM distribution given by Equation (9) to capture the dependency structure among the random variables X_1, X_2 and S. One can easily verify that the parameters $\theta_{ij}(i = 1, 2\ j = 2, 3, i < j)$ in Equation (9) are the correlation coefficients among X_1, X_2 and S. For a greater detail, see Nelsen [8]. From Corollary 2.1 and Equation (9), $T_1 \geq_{st} T_2$ if and only if

$$F_{X_1}(a)\big(1 + \theta_{13}\overline{F}_{X_1}(a)\overline{F}_S(a)\big) \leq F_{X_2}(a)\big(1 + \theta_{23}\overline{F}_{X_2}(a)\overline{F}_S(a)\big), \tag{10}$$

for all $a > 0$. Note that Equation (10) is free of θ_{12}, the parameter dependency between X_1 and X_2; See Remark 2.1. From Equation (10) we see that

- *If* $X_1 \overset{D}{\equiv} X_2$, *then* $\theta_{23} \geq \theta_{13}$ *if and only if* $T_1 \geq_{st} T_2$. *Note that the sufficient condition is also derived directly by Proposition 2.1.*
- *For* $-1 < \theta_{12} < 1$, *and* $\theta_{13} = \theta_{23} = 0$, *then* $X_1 \leq_{st} X_2$ *if and only if* $T_1 \geq_{st} T_2$. *For* $\theta_{12} = 0$ *the result of Boland et al. [3] is obtained. This result can also be obtained also by Proposition 2.1.*
- *If* $X_1 \leq_{st} X_2$ *and* $\theta_{13} \leq 0 \leq \theta_{23}$ *then* $T_1 \geq_{st} T_2$, *since the FGM copula for* $\theta_{ij} \in [0,1]$ *is PQD and for* $\theta_{ij} \in [-1,0]$ *is NQD (See, Nelsen ,[8], p.188). Then, Corollary 2.1 implies the desired result.*

□

Example 2. *(Gumbel copula) Let*

$$F_{X_1,X_2,S}(x_1,x_2,s) = \exp\{-[(-\ln F_{X_1}(x_1))^\theta + (-\ln F_{X_2}(x_2))^\theta + (-\ln F_S(s))^\theta]\}^{1/\theta}, \quad (11)$$

for $x_1 > 0, x_2 > 0, s > 0$ *and* $\theta \geq 1$. *By Corollary 2.1 and Equation* (11), $T_1 \geq_{st} T_2$ *if and only if, for all* $a > 0$

$$\begin{aligned} \exp \quad & \{-[(-\ln F_{X_1}(a))^\theta + (-\ln F_S(a)^\theta)]\}^{1/\theta} - \exp\{-[(-\ln F_{X_1}(a))]^\theta\}^{1/\theta} \\ \leq \quad & \exp\{-[(-\ln F_{X_2}(a))^\theta + (-\ln F_S(a)^\theta)]\}^{1/\theta} - \exp\{-[(-\ln F_{X_2}(a))]^\theta\}^{1/\theta}. \quad (12) \end{aligned}$$

If $X_1 \leq_{st} X_2$, *Condition* (12) *holds for all* $a > 0$, *then* $T_1 \geq_{st} T_2$, *for all* $\theta \geq 1$. *For special case* $\theta = 1$, *the lifetimes* X_1, X_2 *and* S *are independent.*

□

Example 3. *(Clayton copula) Let*

$$F_{X_1,X_2,S}(x_1,x_2,s) = \left(F_{X_1}(x_1)^{-\theta} + F_{X_2}(x_2)^{-\theta} + F_S(s)^{-\theta} - 2\right)^{1/\theta}, \quad (13)$$

for $x_1 > 0, x_2 > 0, s > 0$ *and* $\theta \geq 0$. *Now from Corollary 2.1 and Equation* (13), *we conclude that* $T_1 \geq_{st} T_2$ *if and only if for all* $a > 0$,

$$F_{X_2}(a) + (F_{X_1}(a)^{-\theta} + F_S(a)^{-\theta} - 1)^{1/\theta} \leq F_{X_1}(a) + (F_{X_2}(a)^{-\theta} + F_S(a)^{-\theta} - 1)^{1/\theta}, \forall a > 0. \quad (14)$$

If $X_1 \leq_{st} X_2$ *and Equation* (14) *holds, then* $T_1 \geq_{st} T_2$ *for all* $\theta \geq 0$.

□

Example 4. *(Mardia Tri-variate Pareto Distribution) Let* (X_1, X_2, S) *follows the tri-variate Mardia Pareto distribution with the joint SF*

$$\overline{F}_{X_1,X_2,S}(x_1,x_2,s) = \left(\frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2} + \frac{s}{\sigma_3} - 2\right)^{-\alpha}, x_i > \sigma_i, i = 1,2, S > \sigma_3, \alpha > 0. \quad (15)$$

Then Equation (4) *simplifies to*

$$\left(\frac{x_1}{\sigma_1} - 2\right)^{-\alpha} + \left(\frac{x_1}{\sigma_1} + \frac{x_3}{\sigma_3} - 2\right)^{-\alpha} \leq \left(\frac{x_2}{\sigma_2} - 2\right)^{-\alpha} + \left(\frac{x_2}{\sigma_2} + \frac{x_3}{\sigma_3} - 2\right)^{-\alpha}. \quad (16)$$

Hence for all $\alpha > 0$, $T_1 \geq_{st} T_2$ *provided that* $X_1 \leq_{st} X_2$.

□

If (X_1, S) and (X_2, S) have the same copula and $X_1 \leq_{st} X_2$, then $T_1 \geq_{st} T_2$.

Proof. According to Theorem 6.B.14 of Shaked and Shantikumar ([10], p. 272), we conclude $(X_1, S) \leq_{st} (X_2, S)$. Then Inequality (1) holds and therefore $T_1 \geq_{st} T_2$. □

In Examples (2)-(4), we saw that $X_1 \leq_{st} X_2$ implies $T_1 \geq_{st} T_2$. This result is also derived by Proposition 2.1, since in these examples, (X_1, S) and (X_2, S) have the same copula. For systems with n components see, Jeddi and Doostparast (2015)[5].

References

[1] Belzunce, F., Martinez-Puertas, H. Ruiz, J.M. (2011). On optimal allocation of redundant components for series and parallel systems of two dependent components, *Journal of Statistical Planning and Inference,* **141**, 3094-3104.

[2] Belzunce, F., Martinez-Puertas, H. Ruiz, J.M. (2013). On allocation of redundant components for systems with dependent components, *European Journal of Operational Research,* **230**, 573-580.

[3] Boland, P.J., EI-Neweihi, E., Porschan, F. (1992). Stochastic order for redundancy allocations in series and parallel systems, *Advances in Applied Probability,* **24**, 161-171.

[4] Hu, T., Wang, Y. (2009). Optimal allocation of active redundancies in r-out-of-n systems. *Journal of Statistical Planning and Inference,* **139**, 3733-3737.

[5] Jeddi, H., Doostparast, M. (2015). Optimal redundancy allocation problems in engineering systems with dependent component lifetimes. *Applied Stochastic Models in Business and Industry,* **31**,710-720.

[6] Lehmann, E. L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics,* **37**, 1137-1153.

[7] Mardia, K. V. (1962). Multivariate Pareto distributions. *The Annals of Mathematical Statistics* **33**, 1008-1015.

[8] Nelsen, R. B. (2006). *An introduction to copulas.* Springer, New York.

[9] Romera, R., Valdes, J.E., Zequeira, R.I., 2003. Active-redundancy allocation in systems. *IEEE Transactions on Reliability* **53**, 313-318.

[10] Shaked, M., Shanthikumar, J. G. (2007). *Stochastic Orders.* Springer-Verlag, New York.

[11] Valdes, J.E., Zequeira, R.I. (2003). On the optimal allocation of an active redundancy in a two-component series system, *Statistics and Probability Letters* **63**, 325-332.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Some information measures for three-state networks

Eftekhar, S. [1] and Zarezadeh, S. [2]

[1,2] Department of Statistics, Shiraz University, Shiraz 71454, Iran

Abstract

In this paper, we give some signature-based expressions for the conditional entropy, joint entropy, Kullback-Leibler information, and mutual information of the lifetime of three-state networks. It is shown that the Kullback-Leibler information, and mutual information between the entrance times of the network into its states depend only on the network structure. The results are examined using a numerical example.

Keywords: In this paper, we give some signature-based expressions for the conditional entropy, joint entropy, Kullback-Leibler information, and mutual information of the lifetime of three-state networks. It is shown that the Kullback-Leibler information, and mutual information between the entrance times of the network into its states depend only on the network structure. The results are examined using a numerical example.

1 Introduction

In network reliability analysis, the stochastic properties of the networks lifetime have been considered as an important problem. A network is defined as a collection of *nodes* and *links* in which some particular nodes in the network are called *terminals*. The components of the network (links or nodes) are subject to failure over time. The *state* of the network may change due to the failure of the components.

Let a network consist of n components having independent and identically distributed (i.i.d) lifetimes $X_1, \ldots, X_n$ with common cumulative distribution function (cdf) F_X and probability density function (pdf) f_X. Assume that T denotes the network lifetime. The

[1]sanaeftekhar@gmail.com
[2]s.zarezadeh@shirazu.ac.ir

information measure of T is equal to $I(T) = -H(T)$ where $H(T)$ is called the Shannon entropy of T and defined as

$$H(T) = -\int f_T(t) \log f_T(t) dt. \tag{1}$$

The Kullback-Leibler (K-L) information divergence between f_T and f_X is another information measure which is defined as

$$K(T : X) = K(f_T : f_X) = \int f_T(t) \log \frac{f_T(t)}{f_X(t)} dt \geq 0. \tag{2}$$

Notice that $K(T : X) = 0$ if and only if $f_T(t) = f_X(t)$ almost everywhere. In fact K-L information is a measure of the inefficiency of assuming that the distribution is f_T when the true distribution is f_X. The entropy and K-L information of the k-out-of-n systems and two-state systems have been studied in [1], [3], [4].

There are many cases in reality that several states are defined for the network. For example, we may have a network with three states: *up*, *partial performance*, and *down*. Let $X_1, \ldots, X_n$ be i.i.d. random variables denoting the components lifetime and $X_{i:n}$ be the i-th ordered component lifetime. If T_1 and T denote the entrance times into partial performance and down state, respectively, then

$$P(T_1 > t_1, T > t) = \sum_{1 \leq i < j \leq n} \sum s_{ij} P(X_{i:n} > t_1, X_{j:n} > t), \quad t_1 > 0, t > 0,$$

where $s_{ij} = P(T_1 = X_{i:n}, T = X_{j:n})$ is the (i, j)-th element of a two-dimensional matrix S with size $n \times n$ called signature matrix; see [2].

This paper is an investigation on the information measures of three-state networks lifetime based on the concept of signature matrix. Some representations for the conditional entropy of T given T_1 and the joint entropy of (T_1, T) are provided. It is also shown that the K-L information and mutual information (as a dependency measure) between T and T_1 are free of the components lifetime and depend only on the network structure. The results are examined using a numerical example.

2 Main results

Consider a network with three states: up state, partial performance, and down state. Let the network have n components with i.i.d. lifetimes $X_1, \ldots, X_n$ such that X_1 has pdf f and cdf F. Further, assume that T_1 and T denote the times of entrance into partial performance and down state, respectively. In this section, we would like to explore some information measures on T_1 and T.

2.1 Predictability of T given T_1

For two random variables X and Y, the conditional entropy of Y given X, denoted by $H(Y|X)$, specifies the amount of information needed to predict the outcome of Y given that the value of the X is known. The conditional entropy of Y given X is defined as

$$H(Y|X) = \int f(x) H(Y|x) dx, \tag{3}$$

where $H(Y|x) = -\int f_{Y|X}(y|x) \log f_{Y|X}(y|x) dy$ is the Shannon entropy of Y conditioned on the X taking a certain value x. The Shannon entropy of joint random variable (X, Y)

is called joint entropy defined as $H(X,Y) = -\int\int f_{X,Y}(x,y)\log f_{X,Y}(x,y)dxdy$. With $s_i^{(1)} = \sum_{j=1}^{n} s_{ij}$, $i = 1,\ldots,n$, $s_j^{(2)} = \sum_{i=1}^{n} s_{ij}$, $j = 1,\ldots,n$, we can show that

$$H(T|t_1) = -\int_{F(t_1)}^{1} g(t_1,v)\log\big(f(F^{-1}(v))g(t_1,v)\big)dv,$$

where

$$g(t_1,v) = \frac{\sum\sum_{1\le i<j\le n} s_{ij}\frac{\Gamma(n+1)}{\Gamma(n-j+1)\Gamma(i)\Gamma(j-i)}F^{i-1}(t_1)[v-F(t_1)]^{j-i-1}(1-v)^{n-j}}{\sum_{i=1}^{n} s_i^{(1)}\frac{1}{B(i,n-i+1)}F^{i-1}(t_1)\bar{F}^{n-i}(t_1)},$$

in which $\Gamma(.)$ and $B(.,.)$ are the gamma and beta functions, respectively. Let $h_{\alpha_0,\alpha_1,\alpha_2}(x_1,x_2)$ denote the pdf of Dirichlet distribution with parameters $(\alpha_0,\alpha_1,\alpha_2)$ given, for $x_1 \ge 0$, $x_2 \ge 0$, $0 \le x_1 + x_2 \le 1$, by

$$h_{\alpha_0,\alpha_1,\alpha_2}(x_1,x_2) = \frac{\Gamma(\sum_{i=0}^{2}\alpha_i)}{\prod_{i=0}^{2}\Gamma(\alpha_i)}(1-\sum_{i=1}^{2}x_i)^{\alpha_0-1}\prod_{i=1}^{2}x_i^{\alpha_i-1}, \tag{4}$$

and $g_{a,b}(x)$ be the pdf of beta distribution defined as

$$g_{a,b}(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, \qquad 0<x<1, a>0, b>0. \tag{5}$$

From (3), it can be shown that

$$H(T|T_1) = H(V_1,V) - \sum_{j=1}^{n} s_j^{(2)}E[\log f(F^{-1}(W_j))] - H(V_1),$$

where (V_1,V) has the joint pdf as $\sum\sum_{1\le i<j\le n} s_{ij}h_{n-j+1,i,j-i}(v_1,v)$ and W_j has pdf $g_{j,n-j+1}(w)$.

Based on the properties of conditional entropy, $H(T|T_1) = 0$ if and only if the value of T is completely determined by the value of T_1. $H(T|T_1) = H(T)$ if and only if T and T_1 are independent. Since $H(T,T_1) = H(T|T_1) + H(T_1)$, it can be written

$$H(T_1,T) = H(V_1,V) - \sum_{i=1}^{n} s_i^{(1)}E[\log f(F^{-1}(W_i))] - \sum_{j=1}^{n} s_j^{(2)}E[\log f(F^{-1}(W_j))].$$

2.2 Dependency between T and T_1

The K-L information divergence between T_1 and T is obtained as

$$K(T:T_1) = \int_0^1 \sum_{j=1}^{n} s_j^{(2)}g_{j,n-j+1}(v)\log\frac{\sum_{j=1}^{n} s_j^{(2)}g_{j,n-j+1}(v)}{\sum_{i=1}^{n} s_i^{(1)}g_{i,n-i+1}(v)}dv,$$

where $g_{a,b}(x)$ is as (5). As seen, $K(T:T_1)$, depends only on the network structure and is free of the distribution of components lifetime.

The mutual information between two random variables X and Y is defined by $M(X,Y) = K(f_{X,Y} : f_X f_Y) \ge 0$. $M(X,Y) = 0$ if and only if two random variables X and Y are independent. Then mutual information may be used as a dependency measure in reliability. It can be shown that for a three-state network,

$$\begin{aligned} M(T,T_1) = &\int_0^1\int_0^v \sum\sum_{1\le i<j\le n} s_{ij}h_{n-j+1,i,j-i}(v_1,v-v_1) \\ &\times\log\frac{\sum\sum_{1\le i<j\le n} s_{ij}h_{n-j+1,i,j-i}(v_1,v-v_1)}{\sum_{j=1}^{n} s_j^{(2)}g_{j,n-j+1}(v)\sum_{i=1}^{n} s_i^{(1)}g_{i,n-i+1}(v_1)}dv_1dv, \end{aligned} \tag{6}$$

where h denotes the pdf of Drichlet distribution given in (4). As seen, $M(T, T_1)$ depends only on the network structure and is free of the stochastic mechanism of components failure. That is, the reduction in the uncertainty of network lifetime, T, due to the knowledge of the lifetime T_1 depends only on the structure of the network.

Example. Consider a network with 10 links, 5 nodes, and 5 terminals depicted in Figure 2 of [2]. Let the links be subject to failure and the links lifetime be i.i.d. with standard exponential distribution. Using the signature matrix of this network presented in [2], it can be numerically seen that $H(T_1) = 0.5162952$, $H(T) = 0.7517297$, $H(V_1, V) = -1.703062$, $H(T|T_1) = 0.1485781$, $H(T_1, T) = 0.6648733$, $K(T : T_1) = 0.4120057$, $M(T, T_1) = 0.603152$.

References

[1] Ebrahimi, N., Soofi, E.S. and Zahedi, H. (2004), Information properties of order statistics and spacings. *IEEE Transactions on Information Theory*, **50**, 177-183.

[2] Gertsbakh, I. and Shpungin, Y. (2012), Stochastic models of network survivability, *Quality Technology & Quantitative Management*, **9**(1), 45-58.

[3] Toomaj, A. and Doostparast, M. (2014), A note on signature-based expressions for the entropy of mixed r-out-of-n systems. *Naval Research Logistics (NRL)*, **61**(3), 202-206.

[4] Toomaj, A. and Doostparast, M. (2015), On the Kullback Leibler information for mixed systems. *International Journal of Systems Science*, (ahead-of-print), 1-8.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Reversed Preservation of NWU(3) Class for Parallel System

Gholami, E. [1] and Salehi, E. [2]

[1,2] Department of Basic Science, Birjand University of Technology, Birjand, Iran

Abstract

In this paper we provide new results for the increasing concave average order and the new worse than used in third order (NWU(3)) under the formation of parallel systems with independent and identical (i.i.d) components.

Keywords: Residual lifetime, Class of life distribution, Stochastic order, Reliability.

1 Introduction

One of the interesting problems in reliability theory is the study on aging properties of a coherent system from aging properties of the components. Many authors have paid their attention to investigate the reversed preservation of aging classes for coherent system, specially series and parallel systems. Li and Yam [7] studied the reversed preservation of negative aging classes and proved that when the lifetime of a parallel system belong to new worse than used in second order (NWU(2)) class, the lifetime of its components are also NWU(2). Reversed preservation properties of decreasing failure rate (DFR), increasing mean residual lifetime (IMRL) and new worse than used in expectation (NWUE) class under parallel system are investigated by Belzunce et al. [3]. Li and Yam [7] and Ahmad et al. [1] showed that when the lifetime of series system is IMRL (DMRL), new worse than used in convex order (NWUC) and new better than used in convex average order (NWUCA) the lifetime of its components are also of the same class. In this regard, one can refer to Li and Zuo [9], Ahmad et al. [2], Li and Qui [8], Kayid et al. [5] and Hazara et al. [4].

[1]

[2]salehi@birjandut.ac.ir

In this note, we will make a discussion on the reversed preservation property of NWU(3) class, and it is proved that, if the life time of parallel system consisting of i.i.d components is NWU(3), then its components lifetimes are also NWU(3).

2 Preliminaries

Let F and $\bar{F}$ denote the cumulative distribution function and reliability function of a non-negative random variable X, respectively. The residual lifetime of X at age $t \geq 0$, denoted by $X_t = \{X - t | X > t\}$, is the remaining life of X given survival at age t. The reliability function of X_t, denoted by $\bar{F}_t$, is

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

In the following, we present some stochastic orders considered in this note. Shaked and Shanthikumar [12] is comprehensive reference on these concepts and their properties.

Definition 1. *Let X and Y be two non-negative random variables with distribution functions F and G, and survival functions $\bar{F}$ and $\bar{G}$, respectively. X is said to be smaller than Y in the*

i) usual stochastic order, denoted by $X \leq_{st} Y$, if for all x, $\bar{F}(x) \leq \bar{G}(x)$;

ii) increasing concave average order, denoted by $X \leq_{icva} Y$, if

$$\int_0^\infty \int_0^x \bar{F}(u) du dx \leq \int_0^\infty \int_0^x \bar{G}(u) du dx.$$

Here, we present the definition of NWU(3) class (see Nofal [10]).

Definition 2. *Let X be a non-negative random variable with distribution function F. F is said to be new worse than used in third order, denoted by NWU(3), if and only if for all $t \geq 0$,*

$$\int_0^\infty \int_0^x \bar{F}(u+t) du dx \geq \bar{F}(t) \int_0^\infty \int_0^x \bar{F}(u) du dx,$$

or equivalently, $X_t \geq_{icva} X$.

3 Main results

In this section we consider a parallel system consisting of n independent and identical components with lifetimes $X_1, X_2, ..., X_n$, where X_i has absolutely continuous distribution function F. The parallel system, consisting of n components, operates if and only if at least a component operates. It is obvious that the lifetime of the system is

$$X_{n:n} = \max\{X_1, X_2, ..., X_n\}.$$

The reliability function of $X_{n:n}$, denoted by $\bar{F}_{n:n}(t)$, is given by

$$\bar{F}_{n:n}(t) = 1 - F^n(t), \qquad \text{for all } t \geq 0.$$

For the proof of Theorem 1 and 2 we will need the following lemmas. (Pellerey and Petakos [11]) For any positive integer n and $t \geq 0$, it holds that

$$(\max\{X_1, X_2, ..., X_n\})_t \leq_{st} \max\{(X_1)_t, (X_2)_t, ..., (X_n)_t\},$$

where $X_1, X_2, ..., X_n$ are i.i.d, non-negative random variables.

We obtain the new result in the below lemma that is auxiliary result to prove the main conclusions. Let $W(x)$ be Lebesgue-Stieltjes measure not necessarily positive. If $\Delta(x)$ is an non-negative decreasing function

$$\int_0^\infty \int_0^t dW(x)dt \geq 0,$$

and for all $t \geq 0$, then

$$\int_0^\infty \int_0^t \Delta(x)dW(x)dt \geq 0.$$

Now we have the following theorems.

Theorem 1. *Let $X_1, X_2, ..., X_n$ be n i.i.d random variables with absolutely continuous distribution function F. If $X_{n:n}$ is NWU(3) then X is also NWU(3).*

Next we present the reversed preservation of the increasing concave average order for the parallel system.

Theorem 2. *Let $X_1, X_2, ..., X_n$ be a set of n i.i.d random variable with distribution function F, and $Y_1, Y_2, ..., Y_n$ be another set of n i.i.d random variable with distribution function G. If $X_{n:n} \leq_{icva} Y_{n:n}$, then $X_i \leq_{icva} Y_i$ for $i = 1, 2, ..., n$.*

References

[1] Ahmad, I. A., Ahmed, A. Elbatal, I. and Kayid, M. (2006). An aging notion derived from the increasing convex ordering: the NBUCA class. *Journal of Statistical Planning and Inference*, **136**, 555-569.

[2] Ahmad, I. A., Kayid, M. and Li, X. (2005). The NBUT class of life distributions, *IEEE Transection on Reliability*, **54**, 396-401.

[3] Belzunce, F., Martinez, H. -P. and Ruiz, J. M. (2007). Reversed preservation properties for series and parallel systems, *Journal of Applied Probability*, **44**, 928-937.

[4] Hazara, N. K., Nanda, A. K. and Shaked, M. (2014). Some aging properties of parallel and series systems with a rondom number of components, *Naval Research Logistics*, **61**, 238-243.

[5] Kayid, M., Diab, L. S., Alzughaibi, A. and Ahmad I. A. (2011). Starshaped ordering of life distributions and its aging properties, *IEEE Transactions On Relaiability*, **60**, 257-262.

[6] Lai, C. D. and Xie, M. (2006). *Stochastic aging and dependence for reliability*, Springer, New York.

[7] Li, X. and Richard, C. M. Y. (2005). Reversed preservation properties of some negative aging conceptions and stochastic orders, *Statistical Papers*, **46**, 65-78.

[8] Li, X. and Qiu, G. (2007). Some preservation results of NBUC aging property with applications, *Statistical Papers*, **48**, 581-594.

[9] Li, X. and Zuo, M. J. (2004). Preservation of stochastic orders for random minima and maxima, with applications, *Naval Research Logistics*, **51**, 332-344.

[10] Nofal, M. Z. (2012). On the class of new better than used of life distributions, *Applied Mathematical Sciences*, **137**, 6809-6818.

[11] Pellerey, F. and Petakos, K. (2002). On closure property of the NBUC class under formation of parallel systems, *IEEE Transaction on Reliability*, **51**, 452-454.

[12] Shaked, M. Shanthikumar, J. G. (2007). *Stochastic orders*, Springer, New York.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Characterization of distributions by expectation of functions of some measures in the reliability

Goodarzi, F. [1] and Amini, M. [2] and Mohtashami Borzadaran, G.R. [3]

[1,2,3] Department of Statistics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran

Abstract

Nanda (2010) characterized some distributions with the help of failure rate and mean residual functions. In this paper, we generalize one of his results and characterize a few distributions by expectation of functions of some measures in the reliability.

Keywords: Characterization, Hazard rate, Mean residual life, Expected inactivity time.

1 Introduction

Let X be a random variable having absolutely continuous distribution function $F(t)$, survival function $\overline{F}(t) = 1 - F(t)$ and probability density function $f(t)$. Then the hazard rate function of X is defined as $r(t) = -\frac{d}{dt}\log\overline{F}(t) = \frac{f(t)}{\overline{F}(t)}$.
Besides, let the random variable X have finite moments of all orders with variance $Var(X) = \sigma^2$ and mean $E(X) = \mu$ so that the coefficient of variation of X is $c = \frac{\sigma}{\mu}$.

An useful reliability measure of X is mean residual life (MRL) which is defined as expectation of the residual life random variable $X_t = (X - t \mid X > t)$, given by

$$e(t) = \frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(x)dx.$$

[1]goodarzi_kh@yahoo.com
[2]m-amini@um.ac.ir
[3]grmohtashami@um.ac.ir

The hazard rate and MRL function are related by

$$r(t) = \frac{1 + e'(t)}{e(t)}. \tag{1}$$

It is well known that $r(t)$ determines the distribution function uniquely and hence $e(t)$ also characterizes the distribution. In addition $\overline{F}(t)$ and $r(t)$ are connected by

$$\overline{F}(t) = \exp\left\{-\int_0^t r(x)dx\right\}. \tag{2}$$

Another measure of interest is the expected inactivity time (EIT). Suppose again that the continuous random variable X denotes the lifetime of the system and assume that the system has failed sometime before t. Then the conditional random variable $X_{(t)} = (t - X|X < t)$ measures the time elapsed since the failure of X given that the system has failed sometime before t. The expectation of $X_{(t)}$ which we denote by $m(t)$ is called the EIT of X and is defined as:

$$m(t) = \frac{1}{F(t)}\left\{-\int_0^t F(x)dx\right\}. \tag{3}$$

In recent years, authors have considered the approach of studying inequalities involving expectations of various functions in reliability theory to characterize distributions. See, for instance, Laurent[3] Nanda [5], Bhattacharjee et al. [1] and kundu and Ghosh 2014.

Here we characterize some distributions through expectation of the functions of failure rate and mean residual life.

2 Main results

It is well known that $E(r(X)) \geq \frac{1}{\mu}$, and the equality holds if and only if X is exponentially distributed; see, [4]. Since, for a nonnegative random variable X, E(1/r(X)) is always the mean, the result of Makino restated by Nanda [5] as $E[\frac{1}{r(X)}] \geq \frac{1}{E(r(X))}$, and the equality holds if and only if X is exponentially distributed. This motivated Nanda to prove the following result, see [5].

Theorem 1. *For any nonnegative random variable X,*

$$E\left(\frac{1}{e(X)}\right) \geq \frac{1}{E(e(X))}, \tag{4}$$

the equality holds if and only if X is exponentially distributed.

Below we state the theorem of Nanda [5], that characterize exponential distribution in terms of $E(Xr(X))$.

Theorem 2. *For any nonnegative random variable X,*

$$E(Xr(X)) \geq \frac{2}{1 + c^2}, \tag{5}$$

the equality holds if and only if X is exponentially distributed.

We now generalize the above theorem and characterize the exponential distribution in terms of $E[X^k r(X)]$, where $k > 0$ is an integer. For any nonnegative random variable X,

$$E[X^k r(X)] \geq \frac{k+1}{E(X^{k+1})}(E(X^k))^2, \tag{6}$$

the equality holds if and only if X is exponentially distributed.

Proof. By the Cauchy-Schwarz inequality, we have

$$(E(X^k))^2 \leq \left[\int_0^\infty x^k \overline{F}(x)dx\right]\left[\int_0^\infty \frac{x^k f^2(x)}{\overline{F}(x)}dx\right]. \tag{7}$$

Since

$$\int_0^\infty x^k \overline{F}(x)dx = \frac{E(X^{k+1})}{k+1}, \tag{8}$$

and

$$\int_0^\infty \frac{x^k f^2(x)}{\overline{F}(x)}dx = E[X^k r(X)],$$

(7) reduces to (6). The equality holds if and only if there exists some constant $B(> 0)$ such that, for all $x \geq 0$,

$$\sqrt{\frac{x^k f^2(x)}{\overline{F}(x)}} = B\sqrt{x^k \overline{F}(x)}$$

This gives $r(x)$ =constant, which holds if and only if X is exponentially distributed. □

In the next theorem, we characterize a distribution in which mean residual life is proportional to $1/x$.

Theorem 3. *Let X be an absolutely continuous nonnegative random variable with $E[Xe(X)] < \infty$ and $E\left[\frac{1}{Xe(X)}\right] < \infty$, then*

$$E\left[\frac{1}{Xe(X)}\right] \geq \frac{1}{E[Xe(X)]}, \tag{9}$$

and the equality holds if and only if X follows the distribution with survival function

$$\overline{F}(x) = \frac{x}{a}e^{-(x^2-a^2)/2\theta}, \;\; x > a, a > 0. \tag{10}$$

Proof. By applying the cauchy-Schwarz inequality we obtain (9). The equality in (9) holds if and only if there exists a constant $A(> 0)$ such that

$$\frac{f(x)}{xe(x)} = Axe(x)f(x),$$

which is equivalent to the fact that $e(x) = \theta/x$. Lastly, applying (1) and (2), we get (10). □

Kundu and Ghosh (2014) showed that inequality (9) remain true when MRL is replaced by EIT and equality characterizes the finite range distribution.

References

[1] Bhattacharjee, S., Nanda, A.K. and Misra, S.K., (2013). Inequality involving expectations to characterize distributions. *Statistics and Probability Letters*, **83**, 2113-2118.

[2] Kundu, C. and Ghosh, A., (2014). Inequalities involving expectations of selected functions in reliability theory to characterize distributions. www.arxiv.org/abs/1412.8397.

[3] Laurent, A.G., (1974). On characterization of some distributions by truncation properties. *Journal of the American Statistical Association*, **69**, 823-827.

[4] Makino, T., (1984). Mean hazard rate and its applications to the normal approximation of the Weibull distribution. *Naval Research Logistics Quarterly*, **31**, 1-8.

[5] Nanda, A.K., (2010). Characterizations of distributions through failure rate and mean residual life functions. *Statistics and Probability Letters*, **80**, 752-755.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

A partially accelerated life test planning with competing risks and linear degradation path under tampered failure rate model

Haghighi, F. [1]

[1] Department of Mathematics, Statistics and Computer Science, University of Tehran, Iran

Abstract

In this work, we proposed a partially accelerated life test planning in the presence of competing risks for the products with linear degradation path. The competing risks intensity was considered to be only depended on the degradation value and belong to a parametric family. In the proposed plan, no assumptions are made about failure times distribution and a tampered failure rate model is hold. The maximum likelihood estimation of competing risks intensity parameters as well as the observed Fisher information matrix are derived. A simulation study is conducted to evaluate the performance of the methods and the applicability of the proposed plan is shown by using a real data set.

Keywords: Accelerated life testing, Competing risks, Degradation path , Intensity function, Tempered failure rate model.

1 Introduction

Accelerated life testing (ALT), is commonly used to shorten product's lifetime faster. In such testing, products run at higher level of stress to collect more failure times in a limited test time. Usually variables such as temperature, voltage, pressure, humidity or use rate can be used as accelerating stress to accelerate product's lifetime. Here, we consider an ALT plan in the presence of competing risks when underlying degradation process is linear. Our proposed plan is set as follows: for obtaining failure times in a short period of test a partially ALT are planned and failure times and corresponding competing risk modes are

[1]haghighi@khayam.ut.ac.ir

recorded. These information are extrapolated to estimate the lifetime characteristics at normal conditions through a tampered failure rate model. Tampered failure rate model proposed by Battacharyya and Zanzawi (1989), relates the hazard rate of a unit at one stress level to the hazard rate of that unit at the next stress level. The approach in this modeling is based on the hazard acceleration which seems to have been ignored in the treatment of step-stress ALT despite its wide applications. Using this model and the assumption about linearity of degradation process, the intensities can be expressed in term of degradation at each level of stress. The parameters are estimated by maximum likelihood method and estimated survival function is obtained.

2 Model

We make the following assumptions:

1) Two stress levels S_0 and S_1 $(S_0 < S_1)$ are used. S_0 is normal stress.

2) In each level of stress the degradation path of unit, Z, is an increasing function of time t, and follows a linear model $Z(t) = \frac{t}{A}$, where A is a random vector with distribution function π, and depends on the nature of unit. Linear degradation model has been applied to model the degradation of automobile tire, carbon film, flourescent light buble, plastic substrate AMOLED, etc. See for example, Haghighi and Bae (2015).

3) Competing risks are possible and the failure times due to competing risks assumed to be conditionally independent given $A = a$, and the intensity function corresponding to kth competing risk denoted by λ^k, $k = 1, ..., s$, depends only on degradation level. This is a reasonable assumption for failure due to wear, fatigue or mechanical damages. See, Bagdonavicius *et al.* (2004).

4) A tampered failure rate model is hold to relating the intensities at high level of stress to intensities at lower level of stress. This model is based on the intensity acceleration which have wide use in the reliability as well as survival analysis.

The test is conducted as follows. All test units are initially placed on normal stress S_0, and run until time τ. Then, the stress is changed to high stress S_1, and the test continues until all remaining units fail. Failure time of a unit denotes by $T = min(T^1, ..., T^s)$, where T^k is the failure times corresponding to kth,$k = 1, ..., s$ competing risk. From assumptions 3 and 4, we have

$$\lambda^k(z(t)) = \begin{cases} \lambda_0^k(\frac{t}{a}), & t \leq \tau \\ \lambda_1^k(\frac{t}{a}) = \alpha_k \lambda_0^k(\frac{t}{a}), & t > \tau. \end{cases} \quad (1)$$

where λ_l^k, is the intensity function corresponding to kth competing risk under lth $l = 1, 2$ level of stress.
Denote by $R_l^k(.|A = a)$, the conditional reliability function corresponding to kth comprting risk at lth level of stress. Then,

$$\begin{aligned} R_0^k(t|A = a) &= \mathbf{P}\{T^k > t|A = a, S_0\} \\ &= \exp\{-\int_0^t \lambda_0^k(\frac{w}{a})dw\} \end{aligned}$$

and

$$\begin{aligned} R_1^k(t|A=a) &= R_0^k(\tau|A=a)\mathbf{P}\{T^k > t|A=a, S_1\} \\ &= R_0^k(\tau|A=a)\exp\{-\int_\tau^t \alpha_k \lambda_0^k(\frac{w}{a})dw\}, \\ &= \left[R_0^k(\tau|A=a)\right]^{1-\alpha_k}\left[R_0^k(t|A=a)\right]^{\alpha_k} \end{aligned}$$

From the assumptions that the model is a tampered failure rate the conditional reliability function of a test unit in the presence of competing risks and under simple step-stress test is

$$R^k(t|A=a) = \begin{cases} R_0^k(t|A=a) \\ \text{where} \quad 0 \le t < \tau; \\ \\ R_1^k(t|A=a) = \left[R_0^k(\tau|A=a)\right]^{1-\alpha_k}\left[R_0^k(t|A=a)\right]^{\alpha_k} \\ \text{where} \quad \tau \le t < \infty. \end{cases} \tag{2}$$

3 Estimation

Consider n units are on test and n_1 units fail under S_0 and n_2 units fail under S_1. Let V denote the failure mode of a unit.

$$V = \begin{cases} 1, & \text{if} \quad T = T^1, \\ 2, & \text{if} \quad T = T^2, \\ \vdots & \qquad \vdots \\ s, & \text{if} \quad T = T^s. \end{cases} \tag{3}$$

Here, we focused on the case in which the intensity function is considered to be

$$\lambda^k(z,\theta,\nu) = (\tfrac{z}{\theta_k})^{\nu_k}, \quad k = 1, ..., s. \tag{4}$$

This form of intensity is commonly used in the analysis of wear data. See, Bagdonavicius (2004). The MLEs of parameters satisfy in following equations

$$\hat{\alpha}_k[\hat{\nu}_k] = \left(\frac{\sum_{i=1}^{n_1} t_i(\frac{t_i}{a_i})^{\hat{\nu}_k} + \tau\sum_{i=1}^{n_2}(\frac{\tau}{a_i})^{\hat{\nu}_k}}{\sum_{i=1}^{n_2} t_i(\frac{t_i}{a_i})^{\hat{\nu}_k} - \tau\sum_{i=1}^{n_2}(\frac{\tau}{a_i})^{\hat{\nu}_k}}\right)\left(\frac{\sum_{i=1}^{n_2} I(V_i=k)}{\sum_{i=1}^{n} I(V_i=k)}\right) \tag{5}$$

$$\hat{\theta}_k[\hat{\nu}_k] = \left(\frac{\sum_{i=1}^{n_1} t_i(\frac{t_i}{a_i})^{\hat{\nu}_k} + \tau\sum_{i=1}^{n_2}(\frac{\tau}{a_i})^{\hat{\nu}_k}}{(\hat{\nu}_k+1)\sum_{i=1}^{n} I(V_i=k)}\right)^{\frac{1}{\hat{\nu}_k}} \tag{6}$$

$$g(\hat{\nu}_k) = 0 \tag{7}$$

where

$$\begin{aligned} g(\nu_k) &= \sum_{i=1}^{n} I(V_i=k)\left(\log(\frac{t_i}{a_i}) - \log(\theta_k[\nu_k])\right) + \sum_{i=1}^{n_1}\frac{t^{\nu_k+1}\left(1-(\nu_k+1)\left(\log(\frac{t_i}{a_i}) - \log(\theta_k[\nu_k])\right)\right)}{(\nu_k+1)^2(a_i(\theta_k[\nu_k]))^{\nu_k}} \\ &+ \sum_{i=1}^{n_2}\frac{\tau^{\nu_k+1}\left(1-(\nu_k+1)\left(\log(\frac{\tau}{a_i}) - \log(\theta_k[\nu_k])\right)\right)}{(\nu_k+1)^2(a_i(\theta_k[\nu_k])^{\nu_k}} \end{aligned} \tag{8}$$

Once from (7), $\hat{\nu}_k$, is obtained $\hat{\theta}_k$ and $\hat{\alpha}_k$ are resulted by substituting $\hat{\nu}_k$, in (6) and (5). It is clear that the equation (7), could not be analytically solved and the numerical methods are needed. Simulation studies show that (8) is a decreasing function of the ν and have a unique root. The root of (7), could be easily obtained using "uniroot" in R software.

References

[1] V. Bagdonavicius, F. Haghighi and M. Nikulin, Statistical analysis of general degradation path model and failure time data with multiple failure modes, *Communications in statistics- Theory and methods,* no. 34, pp. 1771-1791, 2005.

[2] G. Bhattacharyya and S. Zanzawi, A tampered failure rate model for step-stress accelerated life test,

statistics- Theory and methods,*Communications in*

statistics- Theory and methods, vol. 5, no. 18, pp. 1627-1643, 1989.

[3] F. Haghighi and SJ. Bae, Reliability estimation from linear degradation and failure time data with competing risks under a step-stress accelerated degradation test, IEEE Trans. Reliability, *no.3 , pp.960-971 , 2015.*

[4] F. Haghighi and M. Nikulin, On the linear degradation model with multiple failure modes, Journal of applied statistics, *vol. 37, no. 9-10, pp. 1499-1509 , 2010.*

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Optimal Design of Bivariate Step Stress Model For Reliability Prediction Under Type II Censoring

Nooshin Hakamipour [1] and Sadegh Rezaei [2]

[1] Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
[2] Faculty of Mathematics and Computer Science, Department of Statistics, Amirkabir University of Technology, Tehran, Iran

Abstract

In this paper, we propose an optimization for the simple step stress accelerated life test for two stress variables under type II censoring. The lifetime of the items follows the Gompertz distribution. Furthermore, we model the effects of changing stress as a cumulative exposure function. By minimizing the asymptotic variance of the maximum likelihood estimator of reliability at time ξ, we obtain the optimal bivariate step stress accelerated life test. Finally, the simulation results are discussed to illustrate the effect of the initial estimates on the optimal values.

Keywords: Bivariate step-stress accelerated life test; Gompertz distribution; Cumulative exposure function; Optimal design.

1 Introduction

Accelerated life tests (ALTs) are used commonly to obtain information quickly on the lifetime of highly reliable products. To implement the SSALT we first apply a low stress to all products, if a product endures the stress (does not fail) we apply a higher stress, if only one change of the stress level is done, it is called a simple step-stress test.

In reliability studies, especially in ALT design, complete data are hard to gather, and the experimenter has to work with censored data.

[1] nooshin.hakami@aut.ac.ir
[2] srezaei@aut.ac.ir

The problem of optimal scheduling of the step stress test has attracted great attention in the reliability literature. Our main objective is to choose the times to change the stress level in such a way that the variance of some estimator of a parameter is minimized under a natural stress level.

Since accelerating just one variable does not result enough failure data, it is desirable to include more stress variables. For instance, in an accelerated life testing of capacitors the two variables temperature and voltage could be used. In addition of resulting more failure data, increasing the number of stress variables would lead to a better understanding of the simultaneous effects of the stress variables.

In this paper, we propose SSALT for two stress variables with Gompertz distribution based on the Type II censoring scheme. The optimum test plan is developed to determine the test duration for each combination of stress levels. The optimization criterion is to minimize the asymptotic variance of the maximum likelihood estimator (MLE) of the life under typical operating conditions with a specified reliability.

2 Bivariate Step stress ALT Model

We consider the SSALT with two stress variables, and each stress variable has two levels. Let S_{lk} be the kth stress level of variable l, where $l = 1, 2$ and $k = 0, 1, 2$. The S_{10}, S_{20} are stress levels at typical operating conditions.

Suppose we have n independent and identically distributed items that are initially put to test at first step with stress levels (S_{11}, S_{21}). The first stress variable is increased from S_{11} to S_{12} at time τ_1. The test is continued until time τ_2, when the other stress variable is increased from S_{21} to S_{22}. The experiment terminates when the predetermined number of failures r is reached. Let n_i be the number of failures at time t_{ij}, where $j = 1, 2, \ldots, n_i$ in step i and $i = 1, 2, 3$.

The basic assumptions are:

1. For any level of stress, the life of test units follows a Gompertz distribution with cumulative distribution function (CDF)

$$F(t) = 1 - \exp\Big\{ - \delta(e^{\theta t} - 1)\Big\}, \quad t \geq 0$$

   where δ is the shape parameter, and θ is the scale parameter.

2. The scale parameter θ_i at test step i, for $i = 1, 2, 3$, is assumed to be a log-linear function of stress levels, and there is no interaction between the two stresses. Thus, we proposed the following life-stress relationship:

$$\begin{aligned} Step1 : \log(\theta_1) &= \beta_0 + \beta_1 S_{11} + \beta_2 S_{21}, \\ Step2 : \log(\theta_2) &= \beta_0 + \beta_1 S_{12} + \beta_2 S_{21}, \\ Step3 : \log(\theta_3) &= \beta_0 + \beta_1 S_{12} + \beta_2 S_{22}. \end{aligned}$$

   where β_0, β_1 , and β_2 are unknown parameters depending on the nature of the product, and the method of test.

3. A cumulative exposure model holds, i.e., the remaining life of a test product depends only on the cumulative exposure it has seen [1].

4. The shape parameter δ is constant for all stress levels.

The CDF of test units under bivariate SSALT and CE model is:

$$G(t) = \begin{cases} 1 - \exp\left\{-\delta\left(e^{\theta_1 t} - 1\right)\right\}, & 0 \le t < \tau_1, \\ 1 - \exp\left\{-\delta\left(e^{\theta_1\tau_1 + \theta_2(t-\tau_1)} - 1\right)\right\}, & \tau_1 \le t < \tau_2, \\ 1 - \exp\left\{-\delta\left(e^{\theta_1\tau_1 + \theta_2(\tau_2-\tau_1) + \theta_3(t-\tau_2)} - 1\right)\right\}, & \tau_2 \le t < \infty. \end{cases} \quad (1)$$

3 Optimality Criterion

The likelihood function from observations t_{ij}, $i = 1, 2, 3$, $j = 1, 2, \ldots, n_i$ and $n-r$ censored items, is obtained from the CDF in equation (1) and the corresponding pdf by replacement in following equation:

$$L(\theta_1, \theta_2, \theta_3, \delta; t) \propto \prod_{j=1}^{n_1} g(t_{1j}) . \prod_{j=1}^{n_2} g(t_{2j}) . \prod_{j=1}^{n_3} g(t_{3j}) . \left[1 - G(t_{(r)})\right]^{n-r}.$$

We can obtain the preliminary estimates of the parameters θ_1, θ_2, θ_3 and δ from previous experiments of similar products, or from a small sample experiment. These estimates are then used to obtain the optimal test design for a censored experiment, but before we proceed we have to define the optimization criterion. Since the mean time to failure is related to the reliability function, the optimization function could also be defined as a function of reliability. In this paper, the optimization criterion is defined to minimize the asymptotic variance of the reliability estimate at time ξ under normal operating conditions.

Let $x_i = (S_{i1} - S_{i0})/(S_{i2} - S_{i0})$, $i = 1, 2$, then $S_{i0} = (S_{i1} - x_i S_{i2})/(1 - x_i)$, $i = 1, 2$. From assumption 2 we can obtain $\log(\theta_0)$ as follows:

$$\log(\theta_0) = \frac{1}{(1-x_1)}\log(\theta_1) + \frac{(x_2 - x_1)}{(1-x_1)(1-x_2)}\log(\theta_2) - \frac{x_2}{(1-x_2)}\log(\theta_3).$$

Thus, the reliability under typical operating conditions at time ξ is

$$\begin{aligned} R_{(S_{10},S_{20})}(\xi) &= \exp\Big\{ -\delta(e^{\theta_0 t} - 1)\Big\}, \\ &= \exp\left\{ -\delta\Big(\exp\left\{ t\ \theta_1^{\frac{1}{(1-x_1)}}\ \theta_2^{\frac{(x_2-x_1)}{(1-x_1)(1-x_2)}}\ \theta_3^{-\frac{x_2}{(1-x_2)}} \right\} - 1\Big)\right\}. \end{aligned}$$

Then the AV of the reliability estimate at time ξ under typical operating conditions can be obtained as follows:

$$AV(\hat{R}_{(S_{10},S_{20})}(\xi)) = H' . F^{-1} . H,$$

where F is the expected Fisher information matrix, which is obtained by taking expectation of the negative of second partial derivatives of $\ell(\theta_1, \theta_2, \theta_3, \delta)$ with respect to θ_1, θ_2, θ_3 and δ. H is the row vector of the first derivative of $\hat{R}_{(S_{10},S_{20})}(\xi)$ with respect to $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\delta}$.

The problem objective is to

$$\text{Minimum AV}\Big[\hat{R}_{(S_{10},S_{20})}(\xi)\Big]$$

Because, the objective function is nonlinear, numerical method is used to solve these above problem. By using the numerical example, we illustrate the calculation of the optimal hold times, and sensitivity analysis.

4 Simulation study

In this section, to examine the effect of changes in the initially estimated parameters, θ_1, θ_2, θ_3, δ and r, on the optimal values of τ_1^* and τ_2^*, sensitivity analysis is performed. This study done by chosing different values for the initial parameters.

Tables 1 present the sensitivity analysis for the different values of the parameter of θ_1, We did a similar examin for r, θ_2, θ_3 and δ. We can conclude, that the optimal values of τ_1^* and τ_2^* slightly increase as parameters increases; and these parameters have a very small effect on the hold times. Therefor, the optimal hold times are not too sensitive.

Table 1: Optimal hold times versus changes in θ_1 with $n = 50, r = 38, \theta_2 = 2.5, \theta_3 = 2, \delta = 0.02$.

2.85		2.95		3.05		3.15	
τ_1^*	τ_2^*	τ_1^*	τ_2^*	τ_1^*	τ_2^*	τ_1^*	τ_2^*
0.7139	0.9035	0.7330	0.9138	0.7569	0.9490	0.7796	0.9599

5 Conclusion

In this paper, we have proposed an optimum design for step-stress accelerated life test with two variables for Gompertz distribution based on Type II censoring. Furthermore, according to simulation studies, we have found that since the optimal hold times are not too sensitive to the model's parameters, thus we anticipate that the proposed design is robust.

References

[1] Khamis, I. H. and Higgins, J. J. (1998), A new model for step-stress testing, *IEEE Transactions on Reliability*, **47**, 131–134.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Evidences in sequential order statistics coming from a general class of distributions

Hashempour, M. [1] Doostparast, M. [2]

[1,2] Department of Statistics, Ferdowsi University of Mashhad, Iran

Abstract

In this paper, statistical evidences in sequential order statistics (SOS) coming from a general class of lifetime distributions, proposed by AL-Hussaini are considered. Weak and misleading evidences are derived in explicit expressions for both simple and composite hypotheses about the parameters of interest and their behaviours with respect to the model parameters are studied in details.

Keywords: sequential order statistics , Hypotheses testing, Likelihood ratio.

1 Introduction

1.1 Statistical Evidence

Following Royall [12], let $\mathtt{ev}\,(H_1, H_2)\,(>0)$ be a given data-based measure of support of the hypothesis H_1 against H_2. Large (Small) values of $\mathtt{ev}\,(H_1, H_2)$ are interpreted as evidence given by data in favor of $H_1(H_2)$. The probabilities of observing strong misleading evidence under H_1 and H_2 are

$$M_1 = P\left(\mathtt{ev}\,(H_1, H_2) < k^{-1} \middle| H_1 \text{ is correct}\right), \tag{1.1}$$

and

$$M_2 = P\left(\mathtt{ev}\,(H_1, H_2) > k \middle| H_2 \text{ is correct}\right), \tag{1.2}$$

[1]ma.hashempour@stu.um.ac.ir
[2]doostparast@math.um.ac.ir

respectively, where k is a known constant greater than unity. The probability of weak evidence under H_i $(i = 1, 2)$ is

$$W_i = P\left(k^{-1} \leq \mathtt{ev}\left(H_1, H_2\right) \leq k \middle| H_i \text{ is correct}\right). \tag{1.3}$$

The natural chooses for k are 3, 8 and 10; See, e.g. Royall [12]. The *decisive* and *correct* evidences are defined by $D_1 = P(\mathtt{ev}\left(H_1, H_2\right) \geq k \middle| H_1$ is correct) and $D_2 = P(\mathtt{ev}\left(H_1, H_2\right) \leq k^{-1} \middle| H_2$ is correct), respectively. In this paper, we consider the likelihood ratio under the hypotheses H_1 and H_2 as a measure of evidence of support for H_1 in favor of H_2. More precisely, let $\Omega_i (i = 1, 2)$ denotes the parameter space under the hypothesis H_i. If Ω_i contains a single point, say θ_i, then

$$\mathtt{ev}\left(H_1, H_2\right) = \mathtt{ev}\left(\{\theta_1\}, \{\theta_2\}\right) = \frac{L(\theta_1; \mathbf{x})}{L(\theta_2; \mathbf{x})}, \tag{1.4}$$

1.2 Sequential order statistics

The concept of *sequential order statistics* (SOSs), introduced by Kamps [11], is an extension of the usual order statistics (OSs) and used for modelling lifetimes of *sequential r-out-of-n systems*. Specifically, consider a given r-out-of-n system consisting of n components and $X_1, \cdots, X_n$ denote the corresponding component lifetimes. Then, the system lifetime (T) coincides to the r-th order statistics among $X_1, \cdots, X_n$, denoted by $X_{r:n}$. In the usual r-out-of-n systems, it is assumed that the lifetimes $X_1, \cdots, X_n$ are independent and identically distributed (i.i.d.) with a common cumulative distribution function (CDF), say F, and denoted by $X_1, \cdots, X_n \overset{i.i.d.}{\sim} F$. Notice that in these systems failing a component does not change distributions of lifetimes of surviving components. Motivated by Cramer and Kamps [2, 3], in practice, the failure of a component may result in higher load on the remaining components and hence causes the distribution of the surviving components change.

In the literature, $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)})$ is called SOSs from $F_1, \cdots, F_n$ and abbreviated by $(X^{\star}_{(1)}, \cdots, X^{\star}_{(n)}) \sim SOS(F_1, \cdots, F_n)$; For other formal definitions and extensions of SOS, see, e.g., Cramer and Kamps [6-9] and Esmailian and Doostparast [7]. Suppose that we observed s independent SOS samples. The available data may be represented as

$$\mathbf{x} = \begin{bmatrix} x_{11} & \dots & x_{1r} \\ \vdots & \ddots & \vdots \\ x_{s1} & \dots & x_{sr} \end{bmatrix}, \tag{1.5}$$

where the i-th row of the matrix $\mathbf{x}$ in (1.5) denotes the SOS sample coming from the i-th population. Let $X^{\star}_{(1)}, \cdots, X^{\star}_{(n)}$ be the first r SOS. Then, the likelihood function (LF) of the available data given by (1.5) is then (Cramer and Kamps [3])

$$L(F_1, \cdots, F_r; \mathbf{x}) = A^s \prod_{i=1}^{s} \left(\prod_{j=1}^{r-1} \left[f_j(x_{ij}) \left(\frac{\bar{F}_j(x_{ij})}{\bar{F}_{j+1}(x_{ij})} \right)^{n-j} \right] f_r(x_{ir}) \bar{F}_r(x_{ir})^{n-r} \right), \tag{1.6}$$

where $A = n!/(n-r)!$ and $\bar{F}(x) = 1 - F(x)$, for $x > 0$.

The problem of estimating parameters on the basis of SOS has been considered in literature. For example, Cramer and Kamps [2] considered the problem of estimating parameters on the basis of s independent multiple SOSs samples under a conditional proportional hazard rates (CPHR) model, defined by $\bar{F}_j(t) = \bar{F}_0^{\alpha_j}(t)$ for $j = 1, \cdots, r$. Here

$F_0(t)$ is a baseline distribution and the hazard rate function of the CDF F_j, defined by $h_j(t) = f_j(t)/\bar{F}_j(t)$ for $t > 0$ and $j = 1, \cdots, n$, is proportional to the hazard rate function of the baseline CDF F_0, i.e. $h_j(t) = \alpha_j h_0(t)$, for $t > 0$.

1.3 An Overview

Statistical evidences in various types of data have been discussed; See for example, De Santis [4], Doostparast and Emadi [5, 6] and references therein. According to the authors' knowledge, there is no work in literature on statistical evidences on the basis of SOS samples. Recently, Hashempour and Doostparast [9] considered evidences in SOS under the CPHR model when the baseline population follows the one-parameter exponential distribution. Here, we consider a wide class of lifetime distributions and extend the findings of Hashempour and Doostparast [9].

2 Evidences in The AL-Hussiani's family of distributions

Al-Hussaini [1] proposed a general family of lifetime distributions of the form

$$F(x;\theta) = 1 - \exp\{-K(x;\theta)\}, \ x > 0, \tag{2.7}$$

where $\theta \in \Theta$ is a vector of parameters and Θ is the parameter space. Here, the function $K(x;\theta)$ is an increasing function in x for all $\theta \in \Theta$ and $K(0;\theta) \equiv 0$ and $K(x;\theta)$ tends to infinity as $x \to +\infty$. Various well-known lifetime distributions including Exponential, Weibull and Pareto belong to the Al-Hussaini's family; For a greater detail, see Al-Hussaini [1]. Under the CPHR model and assuming that the baseline CDF of the parent population belongs to the Al-Hussaini's family with the CDF (2.7), the LF (1.6) simplifies to

$$L(\theta, \boldsymbol{\alpha}; \mathbf{x}) = C \left(\prod_{j=1}^{r} \alpha_j \right)^s \left(\prod_{i=1}^{s} \prod_{j=1}^{r} \frac{\partial K(x_{ij};\theta)}{\partial x_{ij}} \right) \exp\Big\{ - \sum_{i=1}^{s} \sum_{j=1}^{r} m_j K(x_{ij};\theta) \Big\}, \tag{2.8}$$

where $C = (n!/(n-r)!)^s$, $\boldsymbol{\alpha}= (\alpha_1, \cdots, \alpha_r)$ and $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1}$, for $j = 1, \cdots, r$, with convention $\alpha_{r+1} \equiv 0$. Generally, the maximum likelihood estimate (MLE) of the parameter vector θ is derived by maximizing the LF (2.8) for a given structural form $K(.;.)$. Now, we restrict ourselves to a subclass of the AL-Hussaini's family in which one may obtain explicit expressions for the MLE of the parameter vector θ. More precisely, assume that

$$F(x;\theta) = 1 - \exp\{-g(\theta)h(x)\}, \ x > 0, \tag{2.9}$$

where $g(.)$ is a non-negative function and $h(x)$ is an increasing function and $h(0) = 0$ and $h(x) \to +\infty$ as x goes to infinity. This subclass, denoted by $\mathcal{C}$, consists some well-known lifetime distributions such as Exponential, Weibull and Pareto models. From (2.9), the LF (2.8) reads

$$L(\theta, \boldsymbol{\alpha}; \mathbf{x}) = \eta(\mathbf{x};\alpha) g(\theta)^{sr} \exp\{-g(\theta)\xi(\mathbf{x};\alpha)\}, \tag{2.10}$$

where $\xi(\mathbf{x};\alpha) = \sum_{i=1}^{s} \sum_{j=1}^{r} m_j h(x_{ij})$ and

$$\eta(\mathbf{x};\alpha) = \left(\frac{n!}{(n-r)!} \right)^s \left(\prod_{j=1}^{r} \alpha_j \right)^s \left(\prod_{i=1}^{s} \prod_{j=1}^{r} \frac{\partial h(x_{ij})}{\partial x_{ij}} \right).$$

For $g(\theta)=\theta$ and $h(x)=x$, our findings in this paper reduce to the results of Hashempour and Doostparast [9].
One can see from (2.10) that

$$2g(\theta)\xi(\mathbf{x};\alpha)\sim\chi_{2rs}, \tag{2.11}$$

where χ_ν calls for the chi-square distribution with ν degrees of freedom. Therefore, an equi-tail $100(1-\gamma)\%$ confidence interval for $g(\theta)$, when the parameter vector $\boldsymbol{\alpha}$ is known, is

$$\left(\frac{\chi_{2rs,\gamma/2}}{2\xi(\mathbf{x};\alpha)}\quad,\quad\frac{\chi_{2rs,1-\gamma/2}}{2\xi(\mathbf{x};\alpha)}\right), \tag{2.12}$$

where $\chi_{\nu,\gamma}$ stands for the γ-th percentile of the χ_ν-distribution.
If $g(\theta)$ is an increasing function and θ has one-dimension, an equi-tail $100(1-\gamma)\%$ confidence interval for θ is

$$\left(g^{-1}\left(\frac{\chi_{2rs,\gamma/2}}{2\xi(\mathbf{x};\alpha)}\right)\quad,\quad g^{-1}\left(\frac{\chi_{2rs,1-\gamma/2}}{2\xi(\mathbf{x};\alpha)}\right)\right), \tag{2.13}$$

where $g^{-1}(x)$ is the inverse function of $g(x)$.
In sequel, we consider evidences in the available data (1.5) for the problem of hypotheses testing

$$H_1: g(\theta)=g(\theta_1)\ \ v.s\ \ H_2: g(\theta)=g(\theta_2) \tag{2.14}$$

where θ_1 and θ_2 are known constants and $0<g(\theta_1)<g(\theta_2)$. To do this, Equations (1.4) and (2.10) give the evidence for the hypothesis H_1 in favor of H_2 as

$$\mathsf{ev}\left(\{g(\theta_1)\},\{g(\theta_2)\}\right)=\frac{L(\theta_1,\boldsymbol{\alpha};\mathbf{x})}{L(\theta_2,\boldsymbol{\alpha};\mathbf{x})}=\left(\frac{g(\theta_1)}{g(\theta_2)}\right)^{sr}\exp\left\{\left(g(\theta_2)-g(\theta_1)\right)\xi(\mathbf{x};\alpha)\right\}. \tag{2.15}$$

The probabilities of misleading and weak evidences on the basis of s independent SOS samples under the CPHR model with the baseline CDF (2.9) are

$$M_1^{[g]}=F_{\chi_{2rs}}\left(\frac{2g(\theta_1)}{(g(\theta_2)-g(\theta_1))}\ln\left(\frac{\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}}{k}\right)\right), \tag{2.16}$$

$$M_2^{[g]}=1-F_{\chi_{2rs}}\left(\frac{2g(\theta_2)}{(g(\theta_2)-g(\theta_1))}\ln\left(k\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}\right)\right), \tag{2.17}$$

$$\begin{aligned}W_1^{[g]}=&F_{\chi_{2rs}}\left(\frac{2g(\theta_1)}{(g(\theta_2)-g(\theta_1))}\ln\left(k\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}\right)\right)-\\&F_{\chi_{2rs}}\left(\frac{2g(\theta_1)}{(g(\theta_2)-g(\theta_1))}\ln\left(\frac{\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}}{k}\right)\right),\end{aligned} \tag{2.18}$$

$$\begin{aligned}W_2^{[g]}=&F_{\chi_{2rs}}\left(\frac{2g(\theta_2)}{(g(\theta_2)-g(\theta_1))}\ln\left(k\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}\right)\right)-\\&F_{\chi_{2rs}}\left(\frac{2g(\theta_2)}{(g(\theta_2)-g(\theta_1))}\ln\left(\frac{\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}}{k}\right)\right).\end{aligned} \tag{2.19}$$

The L' Hopital rule implies that, for $i = 1, 2$, $\lim_{g(\theta_2)\to+\infty} M_i^{[g]} = \lim_{g(\theta_2)\to+\infty} W_i^{[g]} = 0$. Under this circumstance, the distance between two baseline populations will increasing as much as possible. Thus, the probabilities of misleading and weak evidences vanish. So, even with few SOS data we can make a statement about the hypotheses. Moreover, $\lim_{g(\theta_2)\to g(\theta_1)^+} M_i^{[g]} = \lim_{g(\theta_2)\to g(\theta_1)^+} \left(1 - W_i^{[g]}\right) = 0$. In this case, the distance between two baseline populations will decreasing as much as possible. So, $M_1^{[g]}$ and $M_2^{[g]}$ vanish while $W_1^{[g]}$ and $W_2^{[g]}$ tend to unity. Hence, one can not make decision based on the available data and needs more SOS samples.

An interesting topic in statistical evidence is determination of the global maximum of the misleading evidences. Here, maximizing $M_1^{[g]}$ in Equation (2.16) is equivalent to maximizing $h(\psi) = \ln\left(\psi^{sr}/k\right)/(\psi - 1)$ where $\psi = g(\theta_2)/g(\theta_1)$ with respect to $\psi > 1$. After some algebraic manipulations, one can see that the global maximization of $h(\psi)$ is derived by solving the non-linear equation $\partial h(\psi)/\partial\psi = 0$, or equivalently $1/\psi + \ln(\psi) = 1 + \ln(k)/sr$. Note that the function $h(\psi)$ is convex and therefore the solution of the mentioned equation is unique. Similar arguments imply the next proposition.

Let $u(t) := t^{-1} + \ln(t) - \ln(k)/(sr) - 1$, for $t > 1$. The points of global maximum of $M_2^{[g]}$ and $M_1^{[g]}$, as a function of $g(\psi)$, are derived as the unique solutions of the non linear equations $u(g(\psi)) = 0$ and $u(1/g(\psi)) = 0$, respectively.
Notice that the *decisive* and *correct* evidences are

$$D_1^{[g]} = 1 - F_{\chi_{2rs}}\left(\frac{2g(\theta_1)}{(g(\theta_2) - g(\theta_1))}\ln\left(k\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}\right)\right), \tag{2.20}$$

and

$$D_2^{[g]} = F_{\chi_{2rs}}\left(\frac{2g(\theta_2)}{(g(\theta_2) - g(\theta_1))}\ln\left(\frac{\left(\frac{g(\theta_2)}{g(\theta_1)}\right)^{sr}}{k}\right)\right). \tag{2.21}$$

respectively. In Equations (2.16)-(2.21) the probabilities are given in terms of the CDF of the χ_ν-distribution.

3 Conclusions

Here, we considered statistical evidences in multiple SOS arising from the Al-Hussaini's family distribution populations. Weak and misleading evidences for simple hypotheses about the population parameter were derived in explicit expressions under the CPHR model. We assumed that the parameter vector $\boldsymbol{\alpha}= (\alpha_1, \cdots, \alpha_r)$ of the CPHR model is known. One can see that the measure $\mathtt{ev}\left(\{\theta_1\}, \{\theta_2\}\right)$ given by (2.15) does not depend on the vector $\boldsymbol{\alpha}$.

References

[1] AL-Hussaini, E.K. (1999), Predicting observable from a general class of distribution, *Journal of Statistical Planning and Inference*, **79**, 79–91.

[2] Cramer, E. and Kamps, U. (1996), Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates. *Annals of the Institute of Statistical Mathematics*, **48**(3), 535–549.

[3] Cramer, E. and Kamps, U. (2001a), Estimation with sequential order Statistics from exponential distributions. *Annals of the Institute of Statistical Mathematics*, *53*(2), 307–324.

[4] De Santis, F. (2004), Statistical evidence and sample size determination for Bayesian hypothesis testing, *Journal of Statistical Planning and Inference*, **34**(124), 121-144.

[5] Doostparast, M. and Emadi, M. (2006), Statistical evidence methodology for model acceptance based on record values, *Journal of the Korean Statistical Society*, **35**(2), 167-177.

[6] Doostparast, M. and Emadi, M. (2014), Evidential inference and optimal sample size determination on the basis of record values and record times under random sampling scheme, *Statistical Methods and Applications*, **23**, 41-50.

[7] Esmailian, M. and Doostparast, M. (2014). Estimation based on sequential order statistics with random removals. *Probability and Mathematical Statistics*,**34**(1), 81–95.

[8] Hashempour, M. and Doostparast, M. (2014), Inference based on multiply sequential order statistics from heterogeneous exponential populations. to appear.

[9] Hashempour, M. and Doostparast, M. (2015), Evidences in lifetimes of sequential r-out-of-n systems and optimal sample size determination for exponential populations. to appear.

[10] Johnson, N.L. and Kotz, S. and Balakrishnan N.(1985), *Continuous Univariate Distributions*, John Wiley, Volume 1. Second Edition.

[11] Kamps, U. (1995), A concept of generalized order statistics, *Journal of Statistical Planning and Inference*, **48**, 1–23.

[12] Royall, R. (1997). *Statistical Evidence: A Likelihood Paradigm*, Chapman and Hall, New York.

[13] Royall, R. (2000), On the probability of observing misleading statistical evidence, *Journal of the American Statistical Association*, **95**, 760–780.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

The Prediction of Future Record and Inter-Record Time for Two Parameters Exponential Distribution

Zakerzadeh, H. [1] and Dastbaravarde, A. [2] and Karami, M. [3]

[1,2,3] Department of Statistics, Yazd University , Iran

Abstract

There are many situations where experimental outcomes are a sequence of record observations. In this article, the problem of predicting, either point or interval, for future record and inter-record time based on lower record are given two parameter exponential distribution . Also, a Bayesian prediction is presented for future record and numerical result is given.

Keywords: Two-parameter Exponential distribution , Lower record, Bayes prediction

1 Introduction

Prediction of future observations on the basis of the past and present knowledge is a fundamental problem of statistics, arising in many contexts and producing varied solutions. A predictor can be either a point or an interval predictor.

In this paper, we assume that the data available for study are lower record statistics. Let $X_1, X_2, X_3, ...$ be a sequence of continuous random variables. X_k is a lower record value if its value is smaller than all preceding values $X_1, X_2, ..., X_{k-1}$. By definition, X_1 is a lower record value. Record statistics arise naturally in many practical problems and in applied fields such as athletic events (Kuper and Sterken[6]), Biology (Krug and Jain[6]), catastrophic loss (Hsieh [5])and ... . In this paper, under the Bayesien framework, record values are used to develop prediction of future record. Prediction of future records has

[1]hzaker@yazd.ac.ir
[2]dastbaravarde@yazd.ac.ir
[3]karami-m-66@yahoo.com

been studied by a numbers of statisticians (See, Ahsanullah [1], Arnold et al. [2],). The rest of this article is organized as follows. In Section 2, the basic two-parameter exponential model described that considered and then explained the form of record data based on which all inferential procedures are developed here. In Section 3, the classical and Bayesian prediction (point or interval) of future record and inter-record time, are presented.

2 Model and form of data

A random variable is said to have a two-parameter exponential distribution, denoted by $X \sim Exp(\mu, \sigma)$, the corresponding probability density function (pdf) is given by

$$f(x;\mu,\sigma) = \frac{1}{\sigma} exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}, \quad x \geq \mu, \quad -\infty < \mu < \infty, \quad \sigma > 0, \tag{2.1}$$

As in Doostparast and Balakrishnan[4], our starting point is a sequence of independent random variables $X_1, X_2, X_3, ...$ drawn from the cdf $F(.)$ in (1). We assume that data may be represented by $(\mathbf{R}, \mathbf{K}) := (R_1, K_1, ..., K_{m-1}, R_m)$ where R_i is the i-th record value meaning new minimum and K_i is the number of trials following the observation of R_i that are needed to obtain a new record value R_{i+1}. Throughout this paper, we denote the observed value of these record data by $(\mathbf{r}, \mathbf{k}) := (r_1, k_1, ..., k_{m-1}, r_m)$. The likelihood function associated with the sequence $(r_1, k_1, ..., k_{m-1}, r_m)$ is given by

$$\begin{aligned} L(\theta;\boldsymbol{r},\boldsymbol{k}) = & f(r_m;\theta) \prod_{i=1}^{m-1} f(r_i;\theta)[1-F(r_i;\theta)]^{k_i-1} I_{(-\infty,r_{m-1})}(r_m), \\ & k_i = 1, ..., \infty, \end{aligned} \tag{2.2}$$

where $r_0 \equiv +\infty$ and $I_A(x)$ is the indicator function of the set A. Hence, the likelihood function associated with $(\boldsymbol{r}, \boldsymbol{k})$ for the two parameter exponential becomes

$$\begin{aligned} L(\mu,\sigma;\boldsymbol{r},\boldsymbol{k}) &= \frac{e^{-\frac{r_m-\mu}{\sigma}} e^{-\frac{1}{\sigma}\sum_{i=1}^{m-1} k_i(r_i-\mu)}}{\sigma^m}, \\ & \sigma > 0, \quad -\infty < \mu < r_m < r_{m-1} < ..... < r_1 < \infty \end{aligned} \tag{2.3}$$

3 Prediction

Suppose that we observe only $\{r_1, k_1, ..., r_m\}$ and the goal is to predict either point or interval, for the (m+1)th lower record value and mth inter record time. The conditional distribution of (R_{m+1}, K_m) given $(\boldsymbol{R}, \boldsymbol{K})$ is just the distribution of (R_{m+1}, K_m) given $R_m = r_m$ due to the well-known Markovian property of record statistics. It follows that the predictor density is given by

$$\begin{aligned} f^*(r_{m+1}, k_m \mid \boldsymbol{r}, \boldsymbol{k}; \mu, \sigma) &= \frac{\frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)} e^{-\frac{1}{\sigma}\sum_{i=1}^{m} k_i(r_i-\mu)}}{e^{-\frac{1}{\sigma}(r_m-\mu)} e^{-\frac{1}{\sigma}\sum_{i=1}^{m-1} k_i(r_i-\mu)}} \\ &= \frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)} e^{-\frac{1}{\sigma}(k_m-1)(r_m-\mu)}, \\ & \mu < r_{m+1} < r_m, \quad k_m = 1, ..., \infty. \end{aligned}$$

3.1 Classical prediction

There is a type of classical point prediction where calculated the marginal predictive density, then the predictive estimator r_{m+1} and k_m under square error loss function, introduce $E(R_{m+1})$ and $E(K_m)$. For this purpose, we use the marginal predictive density R_{m+1} and K_m.

$$\begin{aligned} f^*(r_{m+1} \mid \boldsymbol{r}, \boldsymbol{k}; \mu, \sigma) &= \sum_{k_m=1}^{\infty} \frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)} e^{-\frac{1}{\sigma}(k_m-1)(r_m-\mu)} \\ &= \frac{\frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)}}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}}, \quad \mu < r_{m+1} < r_m, \end{aligned}$$

note that, $R_{m+1} \mid (\boldsymbol{r}, \boldsymbol{k})$ has a truncated two-parameters exponential distribution in r_m. If (μ, σ) are known, under square error loss, classical prediction is $E(R_{m+1} \mid \boldsymbol{r}, \boldsymbol{k})$.

$$\begin{aligned} E(R_{m+1} \mid \boldsymbol{r}, \boldsymbol{k}; \mu, \sigma) &= \int_{\mu}^{r_m} r_{m+1} \frac{\frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)}}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}} \, dr_{m+1} \\ &= \frac{1}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}} \int_{\mu}^{r_m} \frac{r_{m+1}}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)} \, dr_{m+1} \\ &= \frac{\mu + \sigma - (r_m + \sigma) e^{-\frac{1}{\sigma}(r_m-\mu)}}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}}, \end{aligned}$$

With using cdf of R_{m+1} can obtain the construction of exact interval prediction for a future record.

$$F^*{}_{R_{m+1}}(t \mid \boldsymbol{r}, \boldsymbol{k}; \mu, \sigma) = \int_{\mu}^{t} \frac{\frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)}}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}} \, dr_{m+1} = \frac{1-e^{-\frac{1}{\sigma}(t-\mu)}}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}}, \quad \mu < t < r_m$$

So, a $100(1-\alpha)$ percent interval prediction for R_{m+1} is given by

$$P(L < R_{m+1} < U \mid (\boldsymbol{r}, \boldsymbol{k})) = 1 - \alpha,$$

so, the lower bound and upper bound are

$$L = \mu - \sigma Ln(1 - \frac{\alpha}{2}(1 - e^{-\frac{1}{\sigma}(r_m-\mu)})), \quad U = \mu - \sigma Ln(1 - (1 - \frac{\alpha}{2})(1 - e^{-\frac{1}{\sigma}(r_m-\mu)})).$$

The marginal predictive density function of K_m given $(\boldsymbol{r}, \boldsymbol{k})$ is given by

$$\begin{aligned} f^*(k_m \mid \boldsymbol{r}, \boldsymbol{k}; \mu, \sigma) &= \int_{\mu}^{r_m} \frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)} e^{-\frac{1}{\sigma}(k_m-1)(r_m-\mu)} \, dr_{m+1} \\ &= [e^{-\frac{1}{\sigma}(r_m-\mu)}]^{k_m-1} (1 - e^{-\frac{1}{\sigma}(r_m-\mu)}), \quad k_m = 1, ..., \infty \end{aligned}$$

given $(\boldsymbol{r}, \boldsymbol{k})$, K_m has Geometric$(1-e^{-\frac{1}{\sigma}(r_m-\mu)})$, If (μ, σ) are known, under square error loss, classical prediction is $E(K_m \mid \boldsymbol{r}, \boldsymbol{k})$ of K_m is

$$\hat{K}_m = E(K_m \mid (\boldsymbol{r}, \boldsymbol{k})) = [\frac{1}{1-e^{-\frac{1}{\sigma}(r_m-\mu)}}],$$

With using cdf of K_m can obtain the construction of exact interval prediction for the future inter-record time,

$$F^*{}_{K_m}(k \mid (\boldsymbol{r}, \boldsymbol{k}); \mu, \sigma) = 1 - e^{-\frac{k}{\sigma}(r_m-\mu)}$$

So, a $100(1-\alpha)$ percent interval prediction for K_m is given by

$$P(L < K_m < U \mid (\boldsymbol{r}, \boldsymbol{k})) = 1 - \alpha,$$

therefore

$$L = [1 - \frac{\sigma Ln(1-\frac{\alpha}{2})}{(r_m - \mu)}], \quad U = [-\frac{\sigma Ln(\frac{\alpha}{2})}{(r_m - \mu)}] \quad \mu < r_m, \quad \sigma > 0.$$

When parameters are unknown, we used the Bayesian method to do on predicting the next record.

3.2 Bayesian prediction

From a strict Bayesian view point, there is clearly no way in which one can say that one prior is better than any other. Presumably one has own subjective prior and must live with all of its lumps and bumps. It is more frequently the case that we elect to restrict attention to a given flexible family of prior distributions and we choose one from the family which seems to best match our personal believes. With this in mind, we consider location and scale parameters are both unknown Here, the joint prior distribution for the parameters μ and σ suggested in the form:

$$\pi(\mu, \sigma) = \pi_1(\mu \mid \sigma)\pi_2(\sigma) \tag{3.4}$$

where

$$\pi_1(\mu \mid \sigma) \propto \frac{1}{\sigma}, \quad \mu \in \mathbb{R} \quad , \tag{3.5}$$

which is the Jeffreys non-informative prior distribution (see Berger [3]) of the parameter μ for fixed value of the parameter σ, and

$$\pi_2(\sigma) = \frac{\beta^\gamma e^{-\frac{\beta}{\sigma}}}{\Gamma(\gamma)\sigma^{\gamma+1}}, \quad \sigma > 0, \quad \gamma > 0, \quad \beta > 0, \tag{3.6}$$

which is the conjugate prior distribution of the parameter σ.
Substituting (3.2) and (3.3) in (3.1), get

$$\pi(\mu, \sigma) = \frac{\beta^\gamma e^{-\frac{\beta}{\sigma}}}{\Gamma(\gamma)\sigma^{\gamma+2}} \quad \sigma > 0, \quad \mu \in \mathbb{R}, \tag{3.7}$$

Hence, the joint posterior distribution of μ and σ is given by

$$\begin{aligned}\pi(\mu, \sigma \mid \boldsymbol{r}, \boldsymbol{k}) = & \frac{(\sum_{i=1}^{m-1} k_i + 1)[\beta + \sum_{i=1}^{m-1} k_i(r_i - r_m)]^{m+\gamma}}{\Gamma(m+\gamma)\sigma^{m+\gamma+2}} \\ & \times exp\left[-\frac{1}{\sigma}\left(\beta + \sum_{i=1}^{m-1} k_i(r_i - \mu)\right)\right], -\infty < \mu \leq r_m, \sigma > 0,\end{aligned} \tag{3.8}$$

where $k_m = 1$.

In this subsection, the problem of point prediction for a future record from a Bayesian approach is considered. In general the Bayes predictive density function of Y given X is given by

$$h(Y \mid X) = \int_{\Theta} f(Y \mid x;\theta)\pi(\theta \mid x)d\,\theta. \tag{3.9}$$

Assume that we have $(r_1, k_1, ..., k_{m-1}, r_m)$, from the $Exp(\mu, \sigma)$. Substituting from (2.3) and (3.5) into (3.6), we get the Bayes predictive density function (r_{m+1}, k_m) given $(\boldsymbol{r}, \boldsymbol{k})$

$$\begin{aligned}
h(r_{m+1}, k_m \mid \boldsymbol{r}, \boldsymbol{k}) &= \int_0^{\infty}\int_{-\infty}^{r_{m+1}} f^*(r_{m+1}, k_m \mid \boldsymbol{r}, \boldsymbol{k}; \mu, \sigma) \times \pi(\mu, \sigma \mid \boldsymbol{r}, \boldsymbol{k})\, d\mu\, d\sigma \\
&= \int_0^{\infty}\int_{-\infty}^{r_{m+1}} \frac{1}{\sigma} e^{-\frac{1}{\sigma}(r_{m+1}-\mu)} e^{-\frac{1}{\sigma}(k_m-1)(r_m-\mu)} \\
&\times \frac{(\sum_{i=1}^{m-1} k_i + 1)[\beta + \eta(\boldsymbol{r}, \boldsymbol{k})]^{m+\gamma}}{\Gamma(m+\gamma)\sigma^{m+\gamma+2}} e\left[-\frac{1}{\sigma}\left(\beta + \sum_{i=1}^{m-1} k_i(r_i - \mu)\right)\right] \\
&\times e^{-\frac{1}{\sigma}(r_m-\mu)}\, d\mu\, d\sigma \\
&= \frac{(\sum_{i=1}^{m-1} k_i + 1)[\beta + \sum_{i=1}^{m-1} k_i(r_i - r_m)]^{m+\gamma}(m+\gamma)}{[\beta + \sum_{i=1}^{m-1} k_i(r_i - r_{m+1}) + k_m(r_m - r_{m+1})]^{m+\gamma+1}(\sum_{i=1}^{m-1} k_i + k_m + 1)},
\end{aligned}$$

The marginal pdf of r_{m+1} given $(\boldsymbol{r}, \boldsymbol{k})$ is given by

$$\begin{aligned}
h(r_{m+1} \mid \boldsymbol{r}, \boldsymbol{k}) &= \sum_{k_m=1}^{\infty} f^*(r_{m+1}, k_m \mid \boldsymbol{r}, \boldsymbol{k}) \\
&= \sum_{k_m=1}^{\infty} \frac{(\sum_{i=1}^{m-1} k_i + 1)[\beta + \sum_{i=1}^{m-1} k_i(r_i - r_m)]^{m+\gamma}(m+\gamma)}{(\sum_{i=1}^{m-1} k_i + k_m + 1)[\beta + \sum_{i=1}^{m} k_i(r_i - r_{m+1})]^{m+\gamma+1}}, \\
&\quad -\infty < r_{m+1} < r_m
\end{aligned} \tag{3.10}$$

Now, the lower and upper $100(1-\alpha)$ prediction bounds for $Y \equiv R_{m+1}$ can be obtained by evaluating the predictive survival function
$Pr(Y \le x \mid (\boldsymbol{r}, \boldsymbol{k}))$ for some positive x. It follows, from (3.7), that

$$\begin{aligned}
H_{R_{m+1}}(x \mid \boldsymbol{r}, \boldsymbol{k}) &= \int_{-\infty}^{x} h(r_{m+1} \mid \boldsymbol{r}, \boldsymbol{k})\, dr_{m+1} \\
&= \sum_{k_m=1}^{\infty} \int_{-\infty}^{x} f^*(r_{m+1}, k_m \mid \boldsymbol{r}, \boldsymbol{k})\, dr_{m+1} \\
&\sum_{k_m=1}^{\infty} \frac{(\sum_{i=1}^{m-1} k_i + 1)[\beta + \sum_{i=1}^{m-1} k_i(r_i - r_m)]^{m+\gamma}}{(\sum_{i=1}^{m-1} k_i + k_m + 1)[\beta + \sum_{i=1}^{m} k_i(r_i - x)]^{m+\gamma}(\sum_{i=1}^{m-1} k_i + k_m)}, \\
&\quad -\infty < x < r_m
\end{aligned}$$

Because r_{m+1} should be less than r_m, so we need only one-sided interval prediction for prediction of r_{m+1} and we choose C to do so

$$P(R_{m+1} > C \mid \boldsymbol{r}, \boldsymbol{k}) = 1 - \alpha \Longrightarrow P(R_{m+1} < C \mid \boldsymbol{r}, \boldsymbol{k}) = \alpha = H_{R_{m+1}}(C \mid \boldsymbol{r}, \boldsymbol{k}). \tag{3.11}$$

The marginal pdf of K_m given $(\boldsymbol{r}, \boldsymbol{k})$ is given by

$$\begin{aligned} g(k_m \mid \boldsymbol{r}, \boldsymbol{k}) &= \int_{-\infty}^{r_m} h(r_{m+1}, k_m \mid \boldsymbol{r}, \boldsymbol{k})\, dr_{m+1} \\ &= \frac{\sum_{i=1}^{m-1} k_i + 1}{(\sum_{i=1}^{m-1} k_i + k_m + 1)(\sum_{i=1}^{m-1} k_i + k_m)}, \qquad k_m = 1, 2, ... \end{aligned}$$

With calculation the cdf of K_m can obtain the construction of exact confidence interval for the future inter-record time.

$$G_{K_m}(k) = \sum_{k_m=1}^{k} \frac{\sum_{i=1}^{m-1} k_i + 1}{(\sum_{i=1}^{m-1} k_i + k_m + 1)(\sum_{i=1}^{m-1} k_i + k_m)}, \tag{3.12}$$

So, a $100(1-\alpha)$ interval prediction for K_m is given by
$P(L < K_m < U) = 1 - \alpha$.

3.3 Illustrative example

In the following, a numerical example is given to illustrate the developed procedures in this section. An lower record sample of size $m = 5$ is generated from the $Exp(1, 5)$ distribution with pdf given by (2.2) and written in Table 1.

Table 1: Generated record sample from $Exp(\mu = 1, \sigma = 5)$

	i				
	1	2	3	4	5
R	1.8508	1.5619	1.1960	1.0517	1.0112
K	3	5	2	4	...

It can be seen that with prior parameters $\gamma = 3$ and $\beta = 4$ in (3.3), by (3.7), we have

$$\begin{aligned} H_{R_{m+1}}(c \mid \boldsymbol{r}, \boldsymbol{k}) &= \sum_{k_m=1}^{\infty} \frac{(\sum_{i=1}^{m-1} k_i + 1)[\beta + \sum_{i=1}^{m-1} k_i(r_i - r_m)]^{m+\gamma}}{(\sum_{i=1}^{m-1} k_i + k_m + 1)[\beta + \sum_{i=1}^{m} k_i(r_i - c)]^{m+\gamma}(\sum_{i=1}^{m-1} k_i + k_m)} \\ &= \sum_{k_m=1}^{\infty} \frac{(\sum_{i=1}^{4} k_i + 1)[4 + \sum_{i=1}^{4} k_i(r_i - r_5)]^{8}}{(\sum_{i=1}^{4} k_i + k_m + 1)(\sum_{i=1}^{4} k_i + k_m)[4 + \sum_{i=1}^{5} k_i(r_i - c)]^{8}} \\ &= \sum_{k_m=1}^{\infty} \frac{1265764157}{(15 + k_m)(14 + k_m)[23.93 - 14c + 1.01k_m - ck_m]^{8}} \\ &= 0.05 \end{aligned}$$

By numerical methods (Maple22) we solve the (3.8):

$$H_{R_{m+1}}(C) = 0.05 \Longrightarrow c = 0.8045$$

So, a 95 percent bayes prediction interval for r_{m+1} is as follows

$$0.8045 < r_{m+1} < 1.0112.$$

Also,

$$P(L < K_m < U) = 0.9$$

By using of G_{K_m} we solve numerically for L and U and give
$45 < K_m < 375$

References

[1] Ahsanullah, M. (2003). *Record values-theory and methods*. Upper Holland.

[2] Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1998). *Records*, John Wiley and Sons, New York.

[3] Berger, J.O. (1980) *Statistical Decision Theory Foundations, Concepts and Methods Probability*. Springer-Verlag, New York Heidelberg Berlin.

[4] Doostparast, M. and Balakrishnan, N. (2010). Optimal record-based statistical procedures for the two-parameter exponential distribution, *Journal of Statistical Computation and Simulation*, **80**, 2002-2019.

[5] Hsieh, P. (2004). A data-analytic method for forecasting next record catastrophe loss, *Journal of Risk and Insurance*, **71**, 309 -322.

[6] Kuper, G. H. and Sterken, E. (2003). Endurance in speed skating: The development of world records, *European Journal of Operational Research*, **148**, 293- 301.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Bayesian estimation for component distribution from system lifetime data

Kelkinnama M. [1]

[1] Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 81744, Iran

Abstract

The problem of Bayesian estimation of the proportionality parameter in the proportional hazard rate model is considered. The Bayes estimate is obtained on the basis of system lifetime data and under the squared error loss function. Explicit forms of Bayes estimator cannot be obtained. Approximate expressions for this estimate are derived using the Simulation-Based method as well as using the Mont-Carlo Integration method. A numerical simulation study is performed to compare the proposed estimates.

Keywords: Bayesian estimate, coherent system, signature.

1 Introduction

In reliability analysis, the reliability estimations of components in a system are often obtained using system life test data. In practice, the exact cause of system failure is sometimes unknown due to various reasons such as the constraints of cost and time. Therefore, the cause of system failure is *masked* and the only observable quantities are the systems-life (failure or censoring time). The estimations on the basis of such data are very useful since they can reflect the actual operational capacity of individual components in system environment.

Component-reliability is often estimated from system life data by using a series system assumption (see, e.g. [5] and [4]). Recently, some authors discussed such inference for a coherent system with known signature, see [1] and [2].

Consider a coherent system consisting of n independent and identically distributed (iid) components with common reliability function $\bar{F}$. It is known that the reliability

[1] m.kelkinnama@cc.iut.ac.ir

function of system can be obtained with the following expression $\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t)$ where, $\mathbf{a} = (a_1, ..., a_n)$ is the minimal signature vector of system.

In proportional hazard rate model we have $\bar{F}(t) = [\bar{G}(t)]^{\alpha}$ where, $\bar{G}(t)$ is the baseline reliability function in $[0, \infty)$, and $\alpha > 0$ is the proportionality parameter. This model includes some common lifetime distributions such as exponential, Pareto and Weibull distributions. Based on the proportional hazard rate model we have,

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i [\bar{G}(t)]^{i\alpha}$$

In the following it is supposed that the $\bar{G}(t)$ is completely specified and an interesting problem is estimating the proportionality parameter, α, based on the system lifetimes when the signature of the system is available. For such model, Ng et al. [1] employed the method of moments, maximum likelihood method and the least squares method. We here consider the problem of estimating α via Bayesian approach.

Suppose m independent n-component systems with the same distribution as T are placed on a life-test and that the corresponding lifetimes $T_1, T_2, ..., T_m$ are observed. The likelihood function of α is

$$L(\alpha) = \alpha^m \prod_{k=1}^{m} \{g(t_k) \sum_{i=1}^{n} i a_i [\bar{G}(t_k)]^{i\alpha - 1}\}$$

where, $g(t)$ is density function of $\bar{G}(t)$.

Under the squared error loss, the Bayes estimator of any function of the unknown parameter and its corresponding minimum Bayes risk, MBR, are the posterior expectation and variance of that function, respectively. Thus to get the Bayes estimate of α and corresponding MBR, we need the first and the second posterior moments. The rth posterior moment of α is obtained as follows

$$E(\alpha^r | t_1, .., t_m) = \frac{\int \alpha^{m+r} \pi(\alpha) \prod_{k=1}^{m} \sum_{i=1}^{n} i a_i [\bar{G}(t_k)]^{i\alpha}\} d\alpha}{\int \alpha^{m} \pi(\alpha) \prod_{k=1}^{m} \sum_{i=1}^{n} i a_i [\bar{G}(t_k)]^{i\alpha}\} d\alpha}$$

where, $\pi(\alpha)$ is the prior distribution of α.

Explicit form of Bayes estimator cannot be obtained. In such case, different approximations are used to establish point estimate.

2 Exponential distribution

We assume that the lifetime of the n components in a system are iid with constant hazard rate, i.e., they are exponentially distributed with reliability $\bar{F}(t) = e^{-\alpha t}$, hence we have $\bar{G}(t) = e^{-t}$.

It is assumed here that the parameter follows the $Gamma(b, c)$ prior distribution, i.e. $\pi(\alpha) = \frac{1}{\Gamma(b)} \alpha^{b-1} c e^{-c\alpha}$ where the hyperparameters b and c are assumed to be non-negative and known. Under the assumptions, the Bayes estimator becomes

$$\tilde{\alpha}_B = \frac{\int_0^{\infty} \alpha^{m+b} e^{-c\alpha} \prod_{k=1}^{m} \sum_{i=1}^{n} i a_i e^{-i\alpha t_k} d\alpha}{\int_0^{\infty} \alpha^{m+b-1} e^{-c\alpha} \prod_{k=1}^{m} \sum_{i=1}^{n} i a_i e^{-i\alpha t_k} d\alpha} \tag{2.1}$$

If the hyperparameters are unknown we can estimate them with empirical Bayes.

Table 1: Bias and MSE of point estimates

		Bias				MSE			
m	Syst. no	MME	MLE	SB	MCI	MME	MLE	SB	MCI
5	1	0.172	0.188	0.087	0.133	0.398	0.390	0.125	0.126
	2	0.162	0.180	0.089	0.132	0.292	0.296	0.114	0.115
10	1	0.080	0.086	0.055	0.955	0.139	0.136	0.082	0.989
	2	0.075	0.083	0.056	0.936	0.096	0.095	0.064	0.953
15	1	0.063	0.066	0.051	2.627	0.066	0.065	0.049	7.054
	2	0.038	0.042	0.029	2.365	0.055	0.054	0.042	5.741

The (2.2) does not take a closed form. Therefore, in the following we adopt two approximation techniques to obtain Bayes estimate. **Simulation-Based method (SB)**

In this method, a random sample from the posterior distribution of the parameter is generated (see [3]). This method is based on generating a random number from prior distribution (say, α_p) and calculating the ratio of likelihoods at this point and at MLE of parameter using the original data. A random number from uniform distribution is generated (say, u); if $u \leq \frac{L(\alpha_p)}{L(\hat{\alpha})}$, then α_p can be considered as a data from posterior distribution. Using simulated random sample from posterior distribution, the Bayes estimate and its MBR are approximated.

Monte Carlo Integration (MCI) method

In computing a complex integral, if the integrand (say $w(x)$) can be decomposed into the product of a function ($v(x)$) and a probability density function ($p(x)$), then the integral can be approximated by drawing a large random sample from the density $p(x)$ and then computing the mean of function $v(x)$ evaluated at generated data.

Using MCI method the (2.2) can be obtained as

$$\tilde{\alpha}_B \cong \frac{\frac{1}{M}\sum_{l=1}^{M}\alpha_l \prod_{k=1}^{m}\sum_{i=1}^{n} i a_i e^{-i\alpha_l t_k}}{\frac{1}{M}\sum_{l=1}^{M}\prod_{k=1}^{m}\sum_{i=1}^{n} i a_i e^{-i\alpha_l t_k}}$$

where, $\alpha_l, l = 1, ..., M$ are random sample from $Gamma(m + b, c)$.

Example 1. *To evaluate the performance of the proposed Bayesian estimate, a large simulation study is carried out. Consider two 4-components coherent systems: series system* $T_1 = \min(X_1, X_2, X_3, X_4)$ *with minimal signature* $\mathbf{a}_1 = (0, 0, 0, 1)$*, and the second system with* $T_2 = \min(X_1, \max(X_2, X_3, X_4))$ *and* $\mathbf{a}_2 = (0, 3, -3, 1)$*. We take* $\alpha = 1$ *and the hyper paramererts* $b = c = 2$*. We generate 1000 sets of system lifetime in order to compare the estimated bias and MSE of point estimators based on i) method of moments (MME) ii) maximum likelihood (MLE) iii) Simulation-Based method (SB) iv) Mont Carlo Integration method (MCI) (for the MME and MLE methods see [1]).*

In this simulation study we observe that the SB method has the smallest bias and the smallest MSE among the other estimators considered here. This means that the Bayes procedure provides better estimates than the method of moments and maximum likelihood approaches in this example. The SB method leads in the better approximation to Bayes estimate of the parameter than the MCI estimate. As it is expected, the MSEs of the different estimators are decreasing with increase the sample size m*. Inference based on samples from the second system results in better estimates than series system (useful when we intend to choose a suitable system structure to run the life-testing experiment).*

References

[1] Ng, H.K.T., Navarro, J. and Balakrishnan, N. (2012), Parametric inference from system lifetime data under a proportional hazard rate model, *Metrika*, **75**, 367388.

[2] Balakrishnan, N., Ng, H.K.T. and Navarro, J. (2011), Linear inference for type-II censored lifetime data of reliability systems with known signatures, *IEEE Transactions on Reliability*, **60**, 426-440

[3] Meeker, W.O., Escobar, L.A. Statistical Methods for Reliability Data, John Wiley Sons, Inc., New York, 1998.

[4] Sarhan, A.M., Kundu, D. (2008), Bayes estimators for reliability measures in geometric distribution model using masked system life test data , *Computational Statistics Data Analysis*, **52**, 1821-1836

[5] Shi, X.L. (2012), Reliability estimations of components in series system using type-II censored and masked life data, *Journal of Physical Sciences*, **16**, 125-132

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

A Note on the reliability of a coherent system in stress-strength model

Khanjari Sadegh, M. [1] and Kasraei, F. [2]

[1] Department of Statistics, Faculty of Mathematical Sciences and Statistics, University of Birjand, Birjand, Iran
[2] University of Payame Noor, Mashhad, Iran

Abstract

In this paper we consider stress-strength modeling for calculating of the stress-strength reliability of a coherent system which is represented as a function of the stress-strength reliabilities of its components. The system components may experience the same or different stress levels. We also point out a mistake result given by Bhattacharya and Roychowdhury (2013) when the components of the system are subjected to a common stress level.

Keywords: Coherent system, Minimal cut sets, Minimal path sets, Stress-strength reliability

1 Introduction

Stress-strength models are important in reliability literature and engineering applications. A system or unit may be subjected to randomly occurring environmental stress such as pressure, temperature and humidity and survival of the system depends on its resistance. In the simplest setup of stress-strength models, a unit functions if its strength is greater than the stress imposed on it. The reliability of the unit is then defined as $R = P(X > Y)$, where X and Y represent the random values of strength of the unit and stress placed on the unit, respectively. The estimation of R has been widely studied under various distributional assumptions on X and Y(see e.g., Kotz et al.(2003)). These models have

[1]mkhanjari@birjand.ac.ir
[2]

also been studied for the systems consist of more than one component. Most of literatures on this subject are concerned when the system components are subjected to a common stress level. Eryilmaz (2008) considered a multivariate stress-strength model for a coherent system. He assumed that the components are subjected to a common random stress. Eryilmaz (2010) expressed the system stress-strength reliability in terms of those of series systems and presented some approximations for system reliability. He assumed that the stress level imposed on the components is common. Eryilmaz (2013) studied the stress-strength reliability of a system with a time dependent(dynamic) strength and a static and common random value of the stress. Bhattacharya and Roychowdhury (2013) studied the stress-strength reliability of a system with different stress levels and claimed their results include the case when the system components are subjected to a common stress level as a special case. In the following section we point out that their claim is not correct and give correct argument.

2 Main result

Let ϕ be the structure function of a coherent system with n components whose random strengths are $X_1, \ldots, X_n$ and suppose the components are subjected to the stress levels $Y_1, \ldots, Y_n$, respectively. The ith component fails if the imposed stress exceeds its strength at any time, i.e. if $Y_i \geq X_i$. Thus $p_i = P(X_i > Y_i)$ gives the stress-strength reliability of the ith component. We define the status of components as follow:

$$Z_i = \begin{cases} 1 & \text{if} \quad X_i > Y_i \\ 0 & \text{if} \quad X_i \leq Y_i \end{cases} \quad i = 1, 2, \ldots, n \tag{2.1}$$

where we assume that $Y_1, \ldots, Y_n$ are independent and Y_i has a continuous distribution function G_i. Also assume that $X_1, \ldots, X_n$ are independent random variables and X_i has a continuous distribution function F_i. We also assume that F_i and G_i are independent distributions. Then the reliability of the coherent system ϕ under the above mentioned stress-strength setup is given by

$$R_\phi = Pr\left\{\phi(Z_1, \ldots, Z_n) = 1\right\}$$

where $\phi(\mathbf{z})$ indicates the state of the system. Note that the binary random variables defined by (2.1) are independent.

In the following using minimal path(cut) sets of the system we obtain a general expression for R_ϕ (for a details on the coherent structures, minimal path(cut) sets etc. see e.g. Barlow and Proschan (1975)). Suppose now that the coherent system has p minimal path sets given by $P_1, \ldots, P_p$ and c minimal cut sets $C_1, \ldots, C_c$. It is known that

$$\phi(\mathbf{z}) = \max_{1 \leq i \leq p} \min_{j \in P_i} z_j = \min_{1 \leq i \leq c} \max_{j \in C_i} z_j$$

$$= 1 - \prod_{i=1}^{p} (1 - \prod_{j \in P_i} z_j) = \prod_{i=1}^{c} \left[1 - \prod_{j \in C_i} (1 - z_j). \right]$$

We have

$$R_\phi = Pr\left\{\cap_{i=1}^{c} [\cup_{j \in C_i} (X_j > Y_j)]\right\} = Pr\left\{\cup_{i=1}^{p} [\cap_{j \in P_i} (X_j > Y_j)]\right\} \tag{2.2}$$

Proof. The first equality was proved by Bhattacharya and Roychowdhury (2013). The second equality can be similarly proved. In the following Remarks we consider some special cases for the Equation (2.2) and also point out the mistake examples given by Bhattacharya and Roychowdhury (2013).

Remark 1. The Equation (2.2) holds true in general even if the independence assumption does not hold. Under independence assumption and according to the form of the minimal cut(path) sets of the system, the first or the second equality in (2.2) may be easier to use than the other. For example in a consecutive-k-out-of-n:F system minimal cut sets are $C_i = \{i, i+1, \ldots, i+k-1\}$, $i = 1, \ldots, c = n-k+1$ which are of simple form and easy to use whereas the minimal path sets of this system do not have such a simple form and also determining of $p(> c)$ for this system is usually complicated. Hence the first equality in (2.2) is easier to use than the second one.

Remark 2. If the minimal cut sets of the system are non-overlapping, the first equality in (2.2) is then reduced to

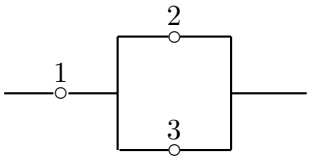
$$R_\phi = \prod_{i=1}^{c} Pr(\cup_{j\in C_i}(X_j > Y_j)).$$

Also when the minimal path sets are disjoint we then have

$$R_\phi = 1 - Pr\left\{\cap_{i=1}^{p}[\cup_{j\in P_i}(X_j \le Y_j)]\right\} = 1 - \prod_{i=1}^{p} Pr(\cup_{j\in P_i}(X_j \le Y_j)).$$

Remark 3. It seems that the situation of common stress level (that is $Y_i = Y, i = 1, \ldots, n$) can be obtained from the case of different stress levels as a particular case. But this is not true in general. Note that when $Y_i = Y$ the binary random variables $Z_1, \ldots, Z_n$ (or equivalently the events $(X_i > Y_i)$) are not independent. For example in a series system we now have $R_\phi = Pr(minZ_i = 1) = Pr(Z_1 = 1, \ldots, Z_n = 1) = Pr(X_1 > Y, \ldots, X_n > Y) = Pr(minX_i > Y) \neq \prod_1^n Pr(X_i > Y)$. Hence those expressions given by Bhattacharya and Roychowdhury (2013) for the reliability of k-out-of-n systems, a series-parallel system(examples 1, 2 and 4), a hi-fi system(Example 5) and for a bridge system(Example 6) are not correct. Her we only give the correct value of the reliability of a series-parallel system.

Example 1. Consider the following series-parallel system.



We have $R_\phi = Pr\left[(X_1 > Y) \cap \{(X_2 > Y) \cup (X_3 > Y)\}\right] =$
$Pr(X_1 > Y) + Pr(X_2 > Y) + Pr(X_3 > Y) - Pr\left[(X_2 > Y) \cap (X_3 > Y)\right]$
$-Pr\left[(X_1 > Y) \cup (X_2 > Y) \cup (X_3 > Y)\right]$ which is (after simplification) equal to

$$Pr(min(X_1, X_2) > Y) + Pr(min(X_1, X_3) > Y) - Pr(min(X_1, X_2, X_3) > Y)$$

REFERENCES

[1] Barlow, R.E. and Proschan, F. (1971), *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, New York.

[2] Bhattacharya, D. and Roychowdhury, S. (2013), Reliability of a coherent system in a multicomponent stress-strength model. *American Journal of Mathematical and Management Sciences*, **32**, 40-52.

[3] Eryilmaz, S. (2008), Multivariate stress-strength reliability model and its evaluation for coherent structures. *Journal of Multivariate Analysis*, **99**, 1878-1887.

[4] Eryilmaz, S. (2010), On system reliability in stress-strength setup. *Statistics and Probability Letters*, **80**, 834-839.

[5] Eryilmaz, S. (2013), On stress-strength reliability with a time-dependent strength. *Journal of Quality and Reliability Engineering*, doi.org/10.1155/2013/417818. Hindawi Publishing.

[6] Kotz, S., Lumelskii, Y. and Pensky, M. (2003), *The Stress-Strength Model and its Generalizations*, World Scientific, Singapore.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

On the Fisher Information of Residual Lifetime

Kharazmi, O. [1] and Asadi, M. [2]

[1,2] Department of Statistics, University of Isfahan

Abstract

Let X be the lifetime of a system and $X_t = (X - t|X > t)$ be the residual lifetime of the system at time t. In the present paper, we propose two dynamic measures to evaluate the dynamic Fisher information (DFI) and dynamic Fisher information distance (DFID) of residual lifetime X_t. We show that DFI and DFID of X_t are connected to well-known reliability measures such as hazard function (HF) and mean residual life (MRL). Several properties of DFI and DFDI are investigated based on aging properties of X_t.

Keywords: Fisher information, Fisher information distance, Mean residual life time , Hazard function

1 Introduction

The Fisher information (FI) and the Fisher information distance (FID) are well known measures with applications in different disciplines such as Statistics, Physics etc (see, for example, Fisher (1925), Johnson(2004) , Frieden (1988) and Frieden (2004)). Given a random variable X with an absolutely continuous density function f, the FI of X (or f) is defined by

$$I(X) = I(f) = \int_{-\infty}^{\infty} \rho^2(x) f(x) dx \tag{1.1}$$

where $\rho(x) = -\frac{f'(x)}{f(x)}$ is the score function corresponding to f. For two random variables X and Y with absolutely continuous density functions f and g, respectively , the DFI

1

2

between f and g is defined by

$$D(f|g) = \int_{-\infty}^{\infty} (\rho_f(x) - \rho_g(x))^2 f(x)dx \tag{1.2}$$

where $\rho_f(x)$ and $\rho_g(x)$ are the score functions corresponding to f and g, respectively.

Duration study is a subject of interest in many branches of science such as reliability, survival analysis, Let X be a nonnegative random variable denoting a duration such as a lifetime of a system with distribution function F, and density function f. Capturing effects of the age t of system on the information about the residual lifetime is important in many applications. For example, in reliability when the system is working at time t, one is interested in the study of the lifetime of the system beyond t. In such case, the set of interest is the *residual lifetime* $S_t = \{x; x > t\}$. Hence the distribution of interest for computing information is the residual distribution with survival function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & x \in S_t \\ 1 & \text{otherwise,} \end{cases} \tag{1.3}$$

where $\bar{F}$ denotes the survival function of X. In the present paper, we introduce a dynamic FI (DFI) and a dynamic FFD (DFID) corresponding to distribution F_t and study its properties in connection to reliability measures such as hazard rate and mean residual life.

2 Dynamic Fisher Information

Assume that $f_t(x)$ and $g_t(x)$ denote the density functions corresponding to $\bar{F}_t(x)$ and $\bar{G}_t(x)$, respectively.

Definition 1. *The DFI of f_t is defined as*

$$\begin{aligned} I_X(t) &= \int_t^{\infty} \left(\frac{\frac{df_t(x)}{dx}}{f_t(x)} \right)^2 f_t(x)dx \\ &= \frac{\int_t^{\infty} \rho^2(x) f(x)dx}{\bar{F}(t)} \end{aligned} \tag{2.4}$$

Definition 2. *The DFID between f_t and g_t is defined as*

$$\begin{aligned} D_t(f||g) &= \int_t^{\infty} (\frac{\frac{df_t(x)}{dx}}{f_t(x)} - \frac{\frac{dg_t(x)}{dx}}{g_t(x)})^2 f_t(x)dx \\ &= \frac{\int_t^{\infty} (\rho_f(x) - \rho_g(x))^2 f(x)dx}{\bar{F}(t)} \end{aligned} \tag{2.5}$$

Clearly for a non-negative random variable X, DFI and DFID reduce, respectively, to FI and FID, when $t \to 0$.

Now we develop some results about DFI and DFID.

Theorem 1. *$I_X(t)$ is increasing (decreasing) if and only if*

$$I_X(t) \leq (\geq) \rho^2(t).$$

Theorem 2. *The survival function $\bar{F}(t)$ can be represented in terms of $I_X(t)$ and $\rho(t)$ as follows:*

$$\bar{F}(t) = e^{-\int_0^t \frac{I_X'(t)}{I_X(t)-\rho^2(t)}}$$

Theorem 3. *If X is a random variable with continuous density f and ϕ is a twice differentiable and invertible function then*

$$I_{\phi(X)}(t) = \frac{1}{\bar{F}(\phi^{-1}(t))} \int_{\phi^{-1}(t)}^{\infty} \frac{1}{\phi'(x)^2} [\frac{f'(x)}{f(x)} - \frac{\phi''(x)}{\phi'(x)}]^2 f(x)dx \tag{2.6}$$

Theorem 4. *Let X and Y have densities f and g, distribution F and G, respectively.*

(a) Assume that g is increasing . $I_{G(X)}(G(t)) \leq \frac{1}{g(t)^2} D_t(f||g)$

(b) Assume that g is decreasing. $I_{G(X)}(G(t)) \geq \frac{1}{g(t)^2} D_t(f||g)$.
In (a) and (b) equalities hold if and only if Y is uniform .

Definition 3. Let X and Y be two random variables with dynamic Fisher information functions $I_X(t)$ and $I_Y(t)$ respectively. X is said to be less than Y in dynamic Fisher information, denoted by $X \leq_{DFI} Y$, if $I_X(t) \leq I_Y(t)$, for all t.

Theorem 5. *Let X and Y have densities f and g, respectively.*

(a) Assume that f is increasing and g is log-convex. If $X \leq_{lr} Y$, then $X \leq_{DFI} Y$.

(b) Assume that f is decreasing and g is log-convex. If $Y \leq_{lr} X$, then $X \leq_{DFI} Y$.

where $\leq_{lr}$ denotes the likelihood ratio order .

3 Connection to reliability functions

Let X be a random variable with density f, survival function $\bar{F}$, the score function $\rho_f(x) = -\frac{f'(x)}{f(x)}$, hazard function $r(x) = \frac{f(x)}{\bar{F}(x)}$ and cumulative hazard function $R(x) = -ln\bar{F}(x)$, respectively.

Theorem 6. *Let X be a random variable with DFI $I_f(x)$. Then*

(a) $E[\rho_f(X)R(X)] = E[r(X)]$, If $\lim_{x\to\infty} f(x) = 0$

(b) $E[\rho_f^2(X)R(X)] = E[I_f(X)]$

(c) $E[\rho_f(X)|X > t] = r(t)$

Theorem 7. *If the smooth conditions $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to 0} f(x) = 0$ are met then we have*

(a) $corr(\rho_f(X), R(X)) = \frac{E[r(X)]}{\sqrt{I(f)}}$

(b) $corr(\rho_f^2(X), R(X)) = \frac{E[I_f(X)]-I(f)}{\sqrt{Var(\rho_f^2(X))}}$

(c) $E[\rho_f(X)R(X)|X > t] = r(t)R(t) + E[r(X)|X > t]$

(d) $E[\rho_f^2(X)R(X)|X > t] = I_f(t)R(t) + E[I_f(X)|X > t]$

(e) $Cov((\rho_f(X), R(X))|X > t) = E[r(X)|X > t] - r(t)$

(f) $Cov((\rho_f^2(X), R(X))|X > t) = E[I_f(X))|X > t] - I_f(t)$

Theorem 8. *Given a random variable X with DFI $I_X(t)$, we have*

(a) $I_X(t) \geq r^2(t)$

(b) $\bar{F}(x) \geq e^{-\int_0^t \sqrt{I_X(x)}dx}$

equalities hold if and only if X is exponential.

Theorem 9. *Let X_e be the equilibrium random variable corresponding to the random variable X. Then the density function of X_e is*

$$f_e(x) = \frac{\bar{F}(x)}{\mu},$$

where μ is the mean of X. The residual Fisher of X_e can be represented as

$$I(X_e) = \frac{E(r(X)|X > t)}{m_X(t)},$$

where $m_X(t)$ denotes the MRL of X.

From above theorem , we particulary get that the Fisher information of X_e is given by $I(X_e) = \frac{E(r(X))}{\mu}$, where μ is the mean of X. Also, from the fact that an IFR (DFR) distribution is DMRL (IMRL), This representation implies that if X is IFR (DFR) then the dynamic Fisher information $I_{X_e}(t)$ is increasing (decreasing) in t.

Theorem 10. *Let X_e be the equilibrium random variable corresponding to the random variable X with density $f_e(x)$ and score function $\rho_{f_e}(x)$, respectively, then*

(a) $E[\rho_{f_e}(X)] = \frac{1}{\mu}$ *, where* $\mu = E(X)$

(b) $E[\rho_{f_e}(X)|X_e > t] = \frac{1}{m_X(t)}$*, where $m_X(t)$ is MRL of variable X.*

(c) $I_{X_e}(t) \geq \frac{1}{m_X^2(t)}$.

(d) $I_{X_e} \leq \frac{\sqrt{I_f}}{\mu}$*, where I_f is FI of random variable X.*

Theorem 11. *Let X_e be the equilibrium random variable corresponding to the random variable X and X_{e^2} be the equilibrium random variable corresponding to the equilibrium random variableX_e then*

$$D_t(f_e||f_{e^2}) = E[(\frac{m_X^{'}(X)}{m_X(X)})^2)|X_e > t]$$

where $m_X(t)$ and D_t are MRL of X and DFID , respectively.

References

[1] Fisher, R. A. (1925),Theory of statistical estimation, *Proc. Cambridge Philos. Soc.*, **22**, 700-725.

[2] Frieden, B. R. (1988), *Physics from Fisher information: a unification*, Cambridge University Press.

[3] Frieden, B. R. (2004), *Science from Fisher information: a unification*, Cambridge University Press.

[4] Johnson, O. (2004), *Information theory and the central limit theorem*, Vol. 8. London: Imperial College Press.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Stress-Strength Model in Discrete Distributions

Khorashadizadeh, M. [1] and Mohtashami Borzadaran, G. R. [2]

[1] Department of Statistics, University of Birjand, Birjand, Iran
[2] Department of Statistics, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract

In this note, using numerical studies and Telescopic form of discrete distributions, we have tried to found a good approximation of stress-strength measure, $R = P(Y < X)$. Also, we have illustrated our method for geometric and discrete Weibull distributions.

Keywords: Reliability function; Stress-Strength model; Telescopic form.

1 Introduction

$R = P(Y < X)$ is a measure of component reliability, which provides a general measure of the difference between two populations and has applications in many areas such as clinical trials, genetics, and reliability. For example, if Y is the response for a control group, and X refers to a treatment group, R is a measure of the effect of the treatment. Or, if Y is the water pressure on the dam wall, and X be the strength of the dam, then the parameter R is of very important in maintenance.

A lot of authors on various topics have done extensive research on stress-strength models and a good review of many papers on theory and applications about $R = P(Y < X)$ can be found on the book by Kotz et al. [3]. We should also mention the recent works of Surles and Padgett [12], Kundu and Gupta [4, 5], Sengupta [10], Kundu and Raqab [6] and Panahi and Asadi [8] which have obtained results due to estimation of $P(X > Y)$ and characterizations related to it in continuous cases.

[1] m.khorashadizadeh@birjand.ac.ir
[2] grmohtashami@um.ac.ir

A few works have been done when the stress (or demand) Y and strength (or supply) X are taken to be discrete random variables. The first works on discrete cases are done by Maiti [7] in the geometric case and Chaturvedi and Tomer [2] and Sathe and Dixit [11] in the negative binomial case. Behboodian [1] find properties of $P(X > Y)$ for binomial distribution.

2 $P(Y < X)$ in Telescopic form of discrete distributions

Recently based on the above method, Rezaei et al. [9] introduced a general form of any discrete distribution and named it *"Telescopic form of distribution"*.

Let X be any non-negative discrete integer valued random variable, then the Telescopic form of its pmf is as follow:

$$p_X(x) = q^{k_\theta(x)} - q^{k_\theta(x+1)}, \quad x = 0, 1, 2, ..., \tag{2.1}$$

where $0 < q < 1$ and $k_\theta(x)$ is a strictly increasing function of x with $k_\theta(0) = 0$ and $k_\theta(\infty) = \infty$.

If $\overline{F}(x) = P(X > x)$ has closed form, then $k_\theta(x)$ can be of form $\frac{\ln \overline{F}(x-1)}{\ln q}$. Also, we have $R_X(x) = P_r(X \geq x) = \sum_{i=x}^{\infty}(q^{k_\theta(i)} - q^{k_\theta(i+1)}) = q^{k_\theta(x)} \quad , x = 0, 1, 2, ..., ..$

If X and Y are two independent discrete random variables with pdf of forms,

$$p_X(x) = q_1^{k_\theta(x)} - q_1^{k_\theta(x+1)}, \quad x = 0, 1, 2, ...,$$

and

$$p_Y(y) = q_2^{k_\theta(y)} - q_2^{k_\theta(y+1)}, \quad y = 0, 1, 2, ...,$$

respectively, we have,

$$\begin{aligned} R = P(Y < X) &= \sum_{x=0}^{\infty} P(Y < x) P_X(x) \\ &= \sum_{x=0}^{\infty} (1 - q_2^{k_\theta(x)})(q_1^{k_\theta(x)} - q_1^{k_\theta(x+1)}) \\ &= 1 - \sum_{x=0}^{\infty} (q_1 q_2)^{k_\theta(x)} + \sum_{x=0}^{\infty} q_1^{k_\theta(x+1)} q_2^{k_\theta(x)}. \end{aligned}$$

As special cases for Geometric distribution ($k_\theta(x) = x$) we have,

$$\begin{aligned} R = P(Y < X) &= 1 - \sum_{x=0}^{\infty} (q_1 q_2)^x + \sum_{x=0}^{\infty} q_1^{x+1} q_2^x \\ &= \frac{q_1(1 - q_2)}{1 - q_1 q_2}, \end{aligned} \tag{2.2}$$

which is the result of Maiti (1995).

As it is seen, in some discrete case, calculation of the R is not always straightforward and the results are not simple. The rest of this note is to study $P(X > Y)$ with a numerical study also.

3 Numerical studies

In discrete distributions of Telescopic form, the problem of computation of $R = P(Y < X)$ is due to calculation of two series $\sum_{x=0}^{\infty}(q_1q_2)^{k_\theta(x)}$ and $\sum_{x=0}^{\infty} q_1^{k_\theta(x+1)}q_2^{k_\theta(x)}$ which have not closed form for some $k_\theta(x)$. So in this section we try to obtain the convergence of function $R(n)$ with respect to n of form below,

$$R(n) = 1 - \sum_{x=0}^{n}(q_1q_2)^{k_\theta(x)} + \sum_{x=0}^{n} q_1^{k_\theta(x+1)}q_2^{k_\theta(x)}.$$

Firstly, note that $R(n)$ is a decreasing function of n and because of the closeness of it values for different values of n, some lines coincide in the following plots.

3.1 Geometric distribution

In Figure **??**, we compute and plot $R(n)$ for different values of q_1 and q_2, and also plot the exact value of R via (2.2).

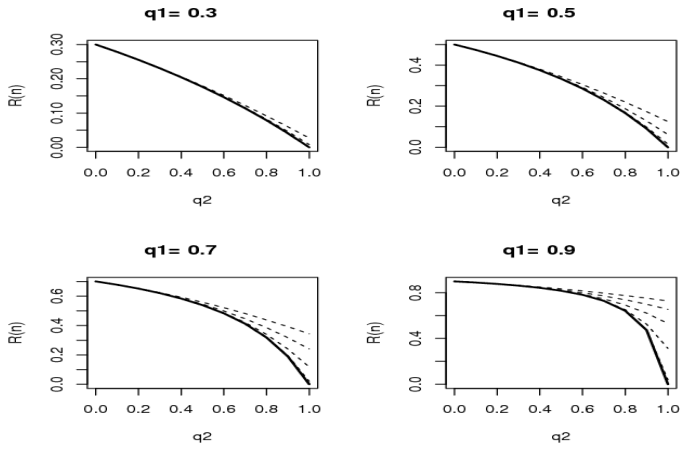


Figure 1: *Computation of exact $R = P(X < Y)$ for two independent geometric distributions with parameters q_1 and q_2 (bold line) and computation of $R(n)$ for different values of parameters and n (from above line to down line the values of n are $2, 3, 5, 10$ and 30 respectively).*

Our computations show that, for n about 20, the differences between $R(n)$ and exact values R for large values of q_2 are negligible and more interesting for small values of q_1 and q_2, the differences are less than 0.00001 for $n = 5$. Although in geometric distribution the exact value of R has obtained, but a good approximation of it can be as follow,

$$\widetilde{R} = q_1 + q_1q_2(q_1 - 1)\left(q_1q_2(q_1q_2 + 1)(q_1^2q_2^2 + 1) + 1\right).$$

3.2 Discrete Weibull distribution

In special case of $k_\theta(x) = x^\theta$ we have discrete Weibull distirbution with pmf $p_X(x) = q^{x^\theta} - q^{(x+1)^\theta}$. Figure **??** shows the computations of $R(n)$ for two independent discrete Weibull distribution with different values of parameters q_1, q_2 and $\theta_1 = \theta_2 = 2$. From numerical study, we found that for small values of q_1 and q_2 and for $\theta > 1$ the convergence

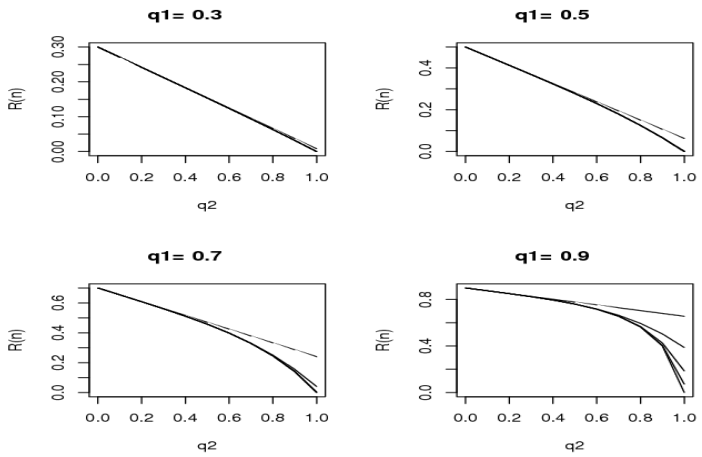


Figure 2: *Computations of $R(n)$ for two independent discrete Weibull distributions with parameters $\theta_1 = \theta_2 = 2$ and different values of parameters q_1 and q_2 for some n (from above line to down line the values of n are $1, 2, 3, 4$ and 100 respectively).*

rate of $R(n)$ is very high for small n and the differences are less than 0.00001. So, we have the following approximation of R in case of $\theta > 1$,

$$\widetilde{R} = q_1 + q_1 q_2 \left[(q_1 q_2)^{2^\theta - 1} (q_2^{1-2^\theta} - 1) + (q_1 q_2)^{3^\theta - 1} (q_2^{2^\theta - 3^\theta} + q_1^{4^\theta - 3^\theta} - 1) + 1 \right].$$

For $\theta < 1$ the differences of $R(n)$ are noticeable for $n < 50$.

References

[1] Behboodian, J.(2003). A look at $P(X > Y)$ in Binomial case. *Presented in fourth Probability and Stochastic Processes Seminar, Gorgan, Iran.*

[2] Chaturvedi, A. and Tomer, S.K. (2002). Classical and Bayesian reliability estimation of the negative binomial distribution. *Jour. Applied Statist. Sci.*, 11, 1, 33-43.

[3] Kotz, S., Lumelskii, Y. and Pensky, M. (2003). *The Stress-Strength Model and Its Generalizations*, World Scientific Press, Singapore.

[4] Kundu, D. and Gupta, R.D. (2005). Estimation of $P[Y < X]$ for generalized exponential distribution. *Metrika*, 61, 3, 291-308.

[5] Kundu, D. and Gupta, R.D. (2006). Estimation of $P[Y < X]$ for Weibull distributions. *IEEE Transactions on Reliability*, 55, 2, 270-279.

[6] Kundu, D. and Raqab, M.Z. (2009). Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution. *Statistics and Probability Letters*, 79, 17, 1839-1846.

[7] Maiti, S.S. (1995). Estimation of $P(X \geq Y)$ in the geometric case. *Journal of the Indian Statistical Association*, 33, 87-91.

[8] Panahi, H. and Asadi, S. (2010). Estimation of $R = P(Y < X)$ for two-parameter Burr Type XII distribution. *World Academy of Science, Engineering and Technology*, 72, 465-470.

[9] Rezaei Roknabadi, A.H., Mohtashami Borzadaran, G.R. and Khorashadizadeh, M. (2009). Some aspects of discrete hazard rate function in Telescopic families. *Economic Quality Control*, 24, 1, 35-42.

[10] Sengupta, S. (2008). Unbiased variance estimation in estimating $P(X > Y)$ for one and two parameter exponential populations. *Metron-International Journal of Statistics*, 66, 3, 329-339.

[11] Sathe, Y.S. and Dixit, U.J. (2001). Estimation of $P[X \leq Y]$ in the negative binomial distribution. *Journal of Statistical Planning and Inference*, 93, 83-92.

[12] Surles, J.G. and Padgett, W.J. (2005). Some properties of a scaled Burr type X distribution.*Journal of Statistical planning and inference*, 128, 1, 271-280.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Estimation of the Shape Parameter of the Weighted Exponential Distribution Under the Record Ranked Set Sampling Plan

Khoshkhoo Amiri, Z. [1] and MirMostafaee, S.M.T.K. [2]

[1,2] Department of Statistics, University of Mazandaran, P. O. Box 47416-1467, Babolsar, Iran

Abstract

In this paper, we discuss the maximum likelihood and Bayes estimation problem for the shape parameter of the weighted exponential distribution based on upper record ranked set samples. A simulation study is presented for the purpose of comparison.

Keywords: Bayesian estimation, maximum likelihood estimation, record ranked set sampling.

1 Introduction

The record ranked set sampling plan has been introduced recently by [2]. Here, we describe this sampling scheme according to [2] as follows: Suppose that we have m independent sequences of continuous random variables. If $R_{i,i}$ denotes the i-th record value in the i-th sequence for $i = 1, ..., m$, then i-th sequence sampling is terminated when $U_{i,i}$ is observed. Then, the only available observations, which are called record ranked set sample (RRSS), include $R_{1,1}, \cdots, R_{m,m}$. Note that $R_{i,i}$'s are independent but not necessarily ordered. These data can be minimal repair times of some reliability systems as mentioned in [2].

The weighted exponential (WE) distribution was introduced by [1] and has the following probability density function (pdf)

$$f(x) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda x}(1 - e^{-\lambda\alpha x}), \quad x > 0, \quad \alpha > 0, \lambda > 0, \tag{1.1}$$

[1] z.khoshkhoo@stu.umz.ac.ir
[2] m.mirmostafaee@umz.ac.ir

where α and λ are the shape and scale parameters, respectively. The WE distribution can be applicable in reliability and therefore estimation of its parameters is important in this field. In this paper, we consider the estimation problem for the shape parameter of the WE distribution based on observed upper RRSSs, when the scale parameter is known. Therefore, without loss of generality, we may assume $\lambda = 1$. Main results as well as a simulation study are given in the next section.

2 Main results

Let $\boldsymbol{U} = (U_{1,1}, ..., U_{m,m})$ be an upper RRSS from the standard WE distribution ($\lambda = 1$), then the likelihood function for the parameter α given $\boldsymbol{u} = (u_{1,1}, \cdots, u_{m,m})$ is (see [2])

$$\begin{aligned} L(\alpha|\boldsymbol{u}) &= \prod_{i=1}^{m} \frac{\{-\log(1-F(u_{i,i}))\}^{i-1}}{(i-1)!}\, f(u_{i,i}) = \left(\frac{\alpha+1}{\alpha}\right)^m e^{-\sum_{i=1}^{n} u_{i,i}} \\ &\times \prod_{i=1}^{m} \frac{1-e^{-\alpha u_{i,i}}}{(i-1)!} \left(-\log\left[\frac{\alpha+1}{\alpha}e^{-u_{i,i}} - \frac{1}{\alpha}e^{-(\alpha+1)u_{i,i}}\right]\right)^{i-1}, \end{aligned} \tag{2.2}$$

and therefore the log-likelihood function is

$$\begin{aligned} \ell &= m\log(\alpha+1) - m\log\alpha - \sum_{i=1}^{m} u_{i,i} + \sum_{i=1}^{m}\log(1-e^{-\alpha u_{i,i}}) - \sum_{i=1}^{m}\log(i-1)! \\ &+ \sum_{i=1}^{m}(i-1)\log\left[u_{i,i} + \log\alpha - \log(\alpha+1-e^{-\alpha u_{i,i}})\right]. \end{aligned}$$

The maximum likelihood estimator (MLE) of α, say $\widehat{\alpha}_M$, can be obtained as the solution of the following non-linear equation

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha+1} - \frac{m}{\alpha} + \sum_{i=1}^{m} \frac{u_{i,i}e^{-\alpha u_{i,i}}}{1-e^{-\alpha u_{i,i}}} + \sum_{i=1}^{m} \frac{(i-1)\left(\frac{1}{\alpha} - \frac{1+u_{i,i}e^{-\alpha u_{i,i}}}{\alpha+1-e^{-\alpha u_{i,i}}}\right)}{u_{i,i} + \log\alpha - \log(\alpha+1-e^{-\alpha u_{i,i}})} = 0.$$

For Bayesian estimation, we take the inverse gamma distribution as the prior density function

$$\pi(\alpha) = \frac{b^c}{\Gamma(c)}\alpha^{-c-1}e^{\frac{-b}{\alpha}}, \tag{2.3}$$

where b and c are the positive hyperparameters. Note that for $b = c = 0$, we arrive at the non-informative prior. The posterior distribution of α is then obtained to be

$$\pi(\alpha|\boldsymbol{u}) = \frac{(\alpha+1)^m e^{\frac{-b}{\alpha}}}{C_0\alpha^{m+c+1}} \prod_{i=1}^{m}(1-e^{-\alpha u_{i,i}})\left(-\log\left[\frac{\alpha+1}{\alpha}e^{-u_{i,i}} - \frac{e^{-(\alpha+1)u_{i,i}}}{\alpha}\right]\right)^{i-1},$$

where
$C_0 = \int_0^\infty \frac{(\alpha+1)^m e^{\frac{-b}{\alpha}}}{\alpha^{m+c+1}} \prod_{i=1}^{m}(1-e^{-\alpha u_{i,i}})\big(-\log\big[\frac{\alpha+1}{\alpha}e^{-u_{i,i}} - \frac{e^{-(\alpha+1)u_{i,i}}}{\alpha}\big]\big)^{i-1}\mathrm{d}\alpha$. Let $\widehat{\alpha}$ be an estimator of α, then the SEL function is defined as $L_1(\alpha, \widehat{\alpha}) = (\widehat{\alpha}-\alpha)^2$. The Bayes estimate of α under SEL function based on RRSS, denoted as $\widehat{\alpha}_S$, is the mean of the posterior density, i.e.
$\widehat{\alpha}_S = \int_0^\infty \frac{(\alpha+1)^m e^{\frac{-b}{\alpha}}}{C_0\alpha^{m+c}} \prod_{i=1}^{m}(1-e^{-\alpha u_{i,i}})\big(-\log\big[\frac{\alpha+1}{\alpha}e^{-u_{i,i}} - \frac{e^{-(\alpha+1)u_{i,i}}}{\alpha}\big]\big)^{i-1}\mathrm{d}\alpha$.

The SEL function is symmetric namely it assigns equivalent dimensions to underestimation

and overestimation. Thus, it may not be appropriate in situations that overestimation and underestimation have different consequences. Here, we consider a well-known asymmetric loss function, called the LINEX loss function, which is defined as

$$L_2(\alpha, \widehat{\alpha}) = p\ [\exp\{q(\widehat{\alpha} - \alpha)\}) - q(\widehat{\alpha} - \alpha) - 1], \quad p > 0, \quad q \neq 0.$$

Without loss of generality, we may assume $p = 1$. The sign and magnitude of parameter q must be determined properly. For positive values of q, the overestimation is more serious than underestimation and vice versa. The Bayes estimate of α under the LINEX function based on RRSS, denoted as $\widehat{\alpha}_L$, is $\widehat{\alpha}_L = -\frac{1}{q} \log E(e^{-q\alpha}|\boldsymbol{u})$, where

$$E(e^{-q\alpha}|\boldsymbol{u}) = \int_0^\infty e^{-q\alpha} \pi(\alpha|\boldsymbol{u}) \mathrm{d}\alpha.$$

Here, we consider the Markov chain Monte Carlo (MCMC) method and the Gibbs sampler to simulate samples from the posterior density and then compute the Bayes estimators of α. The posterior density function of α can be rewritten as $\pi(\alpha|\boldsymbol{u}) \propto g(\alpha|\boldsymbol{u})h(\alpha)$, where $g(\alpha|\boldsymbol{u})$ is the inverse gamma density function with parameters $a + m$ and b and $h(\alpha)$ is given by
$h(\alpha) = (\alpha + 1)^m \prod_{i=1}^m (1 - e^{-\alpha u_{i,i}}) \big(- \log \big[\frac{\alpha+1}{\alpha} e^{-u_{i,i}} - \frac{1}{\alpha} e^{-(\alpha+1)u_{i,i}} \big] \big)^{i-1}$.

Now, using the importance sampling procedure, we can state the following algorithm to approximate the Bayes estimators
Step 1. Generate α_1 from $g(\alpha|\boldsymbol{u})$.
Step 2. Repeat Step 1, N times to obtain $\alpha_1, ..., \alpha_N$.
Step 3. The approximate value for the Bayes estimator of α under the SEL function is $\widehat{\alpha}_{MS} = \sum_{i=1}^N \alpha_i w_i$ and the approximate value for $\widehat{\alpha}_L$ is $\widehat{\alpha}_{ML} = -\frac{1}{q} \log \left(\sum_{i=1}^N e^{-q\alpha_i} w_i \right)$, where $w_i = \frac{h(\alpha_i)}{\sum_{j=1}^N h(\alpha_j)}$.

2.1 A simulation study

In this section, we performed a simulation in order to compare the point estimators. In this simulation, we randomly generated $M = 2000$ upper RRSSs of size $m = 6$ from the WE distribution with $\alpha = 2$. We considered three cases for the prior distribution as follows: **Case I**: A case that is very close to the non-informative prior with $b = 0.01$ and $c = 0$. **Case II**: Informative prior with prior information $E(\alpha) = 2$ =true value, and $Var(\theta) = 2$ and from (2.3), we obtain $b = 6$ and $c = 4$. **Case III**: Informative prior with prior information $E(\alpha) = 2$ and $Var(\theta) = 0.5$ and consequently from (2.3), we have $b = 18$ and $c = 10$. We then obtained the MLEs. In addition, the approximate Bayes estimators of α under SEL and LINEX (for $q = -2, 2$) functions, which are denoted by $\widehat{\alpha}_{MS}(i)$ and $\widehat{\alpha}_{ML}(i)$, in the i-th iteration, respectively, were obtained using the importance sampling procedure with $N = 500$. The estimated risks (ERs) of the estimators were calculated using the relations $ER_S(\widehat{\alpha}_{MS}) = \frac{1}{M} \sum_{i=1}^M [\widehat{\alpha}_{MS}(i) - \alpha]^2$, and

$$ER_L(\widehat{\alpha}_{ML}) = \tfrac{1}{M} \textstyle\sum_{i=1}^M [\exp\{q(\widehat{\alpha}_{ML}(i) - \alpha)\}) - q(\widehat{\alpha}_{ML}(i) - \alpha) - 1].$$

Actually, we calculated the ER of each Bayes estimator according to its own loss function. For the MLEs, we calculated both kinds of ERs, i.e. ER_S and ER_L to compare them with their corresponding approximate Bayes estimators. The results are given in Table 1. From Table 1, we observe that the ERs of the Bayes estimators are smaller than the ERs

Table 1: The results of the simulation.

	ER_S	ER_L	
		$q=-2$	$q=2$
MLE	42.71484	2.222723	>1000
Case I	3.969463	48.78501	3.003302
Case II	0.4819587	1.041379	0.8500697
Case III	0.07773791	0.1794383	0.268095

of the corresponding MLEs (except for one case) which reveals that the Bayesian methods are superior to the likelihood ones. We also see that the smallest ERs belong to Case III that is reasonable as Case III has the smallest prior variance and therefore is the most informative case.

References

[1] Gupta, R.D. and Kundu D. (2009), A new class of weighted exponential distributions, *Statistics*, **43**, 621-634.

[2] Salehi, M. and Ahmadi, J. (2014), Record ranked set sampling scheme, *Metron*, **72**, 351-365.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Estimating the Survival Function Using Different Copula Functions

Mireh, S. [1] and Khodadadi, A. [2]

[1,2] Department of Statistics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran

Abstract

The analysis of reliability and survival functions is one of the most important goals in system safety, especially when several dependent failure modes influence on failure time. In previous research, dependency between the degradation process and traumatic failure time has been studied in limited detail (special closed form expression). This study gives some contributions that evaluate reliability metrics with more than one failure mechanism which may not be independent and possibly follow a different distribution function. We have used different copula functions as a basis to develop a proposal model and analysis methods. Finally, real and simulation data were used to review the suggested approach.

Keywords: Survival Function, Copula Function, Traumatic Failure Time, Degradation Process, General Dependency Structure.

1 Introduction

At the beginning of the work done in reliability, engineers used the survival function to describe the uncertainties about the failure times. From a statistical point of view, accurate estimation of the survival function needs the observation of failure times of many items. Because of advances in materials science, it is not always possible to observe many failures, and even if such failure times are possible to obtain, they are not independent as they all might be affected by a common environment. Thus, it is difficult to obtain the

[1] s_mireh@sbu.ac.ir
[2] a_khodadadi@sbu.ac.ir

accurate reliability estimation based on life data. Another method is to assess the failure of a component based on the characteristics of the process that caused its failure, is often named degradation process. So there are two kinds of data: Hard and Soft failure. Hard failure causes instant product failure, is characterized as both complete and sudden, and cannot be forecasted by prior testing or examination. On the other hand, soft failure, occurs when a measurable physical degradation reaches a critical threshold level (z_0), which is often specified by industrial standards. Soft failures can be predicted by prior testing or examination.

We can speak of a classification of research done according to failure mechanisms (hard failure or soft failure), operating conditions (normal or accelerated stress), performance characteristic dimensionality (univariate or multivariate), and structural dependency (dependent or independent). Many studies has been done in each category (such as [1], [2], [3]). However, no studies considered both types of dependent structures as one model.

Hence, the current study developed a methodology that took into consideration both degradation measurements and observed failure time. The dependency structure was described by different copula functions.

This paper is organized as follows. In section 2, our proposed method is discussed. Examples of real and simulated data are expressed in section 3, followed by the conclusion in section 4.

2 The model

Suppose degradation data ($Z(t)$) and observed failure time (T) are dependent. We define T as the minimum of soft (T^0) and hard failure times (T^1): $T = \min(T^0, T^1)$. The purpose of this section is to estimate the survival function with the assumption that the degradation data and observed failure time are not independent. The copula function is utilized as a basis to develop the proposed model and analysis.

In parameter estimation, we suppose that the increments of degradation and failure time data have gamma and Weibull distributions, respectively. Let $\Theta = (\theta, \alpha_1, \beta_1, \alpha_2, \beta_2)$ be the collection of unknown parameters. The maximum likelihood (ML) method is used for parameter estimation. Given the observed data, the likelihood is:

$$L(\Theta) = \prod_{i=1}^{n} \left(f_T(t_i).g_{Z(t_i)}(z_0).c(F_T(t_i), G_{Z(t_i)}(z_0)) \right)^{\delta_i} . \prod_{i=1}^{n} \left(g_{Z(t_i)}(z_0) \right)^{1-\delta_i},$$

where $c(F_T(t_i), G_{Z(t_i)}(z_0)) = \frac{\partial^2 C(F_T(t_i), G_{Z(t_i)}(z_0))}{\partial F_T(t_i) \partial G_{Z(t_i)}(z_0)}$, c is the copula density, f, F, g and G are the marginal density and distribution function of Weibull and gamma, respectively.

After estimating the parameters, we can calculate the survival function using different copulas as follows:

$$S(t) = v - C(u, v)$$

where C is an copula cdf and $F_T(t) = u$ and $G_{Z(t)}(z_0) = v$. For example for Frank copula we have:

$$S(t) \quad = \quad v + \frac{1}{\theta} \ln \left(1 - \frac{(1 - e^{-\theta u})(1 - e^{-\theta v})}{(1 - e^{-\theta})} \right)$$

3 Examples

3.1 Real data Study

The numerical data are taken from [4]. In the data, 21 samples have been tested for fatigue crack development. To demonstrate the degradation along with traumatic failure time, we use degradation data and simulate the traumatic failure time using the Frank copula ($\theta = -15$). Next, the minimum soft and hard failure times are considered as failure time (T). The item is considered failed if either the crack size exceeds 1.6 in. (δ_i=0) or the failure happened according to the mechanism, which is characterized as a hard failure (δ_i=1). Finally, we can estimate the model parameters.

Table 1 shows these estimations only for Frank copula. In this table, the independence and negative dependency between traumatic failure time and degradation is assumed. Next, we compare these two assumptions. α_1 and β_1 are used to express the parameters of the Weibull distribution and α_2 and β_2 express the gamma parameters.

Table 1: Estimation of parameters using Frank copula

Diag ColumnmnHead II Dependency Parameters	α_1	β_1	α_2	β_2	θ
Negative Dependency	1.723	0.101	41.263	0.406	-203.741
Independency	2.826	0.062	89.198	0.166	-

According to the estimated parameters and the proposal method in section 3, we can estimate the survival function. Our results show that using different copula functions, the two methods (assuming the dependency and independency between T and Z) have different behaviors. The efficiency of the model drops drastically when we do not consider the dependency factor, through the general dependency structure.

3.2 Simulation Study

In this section we simulate $n = 100$ of degradation and failure time data. To evaluate the effect of dependencies between data, we consider several negative dependencies and different copula functions for comparison with each other. After the simulation is completed, the distributions parameters are estimated for several copula functions and different values of the copula parameter, but only $\theta = -200$ for Frank, Rotated BB6, Rotated Gumbel 90 and Gaussian copula is shown in Table 2.

Table 2: Estimated parameters for simulation data

Diag ColumnmnHead II Copula functionParameters	α_1	β_1	α_2	β_2	Copula parameter(s)
Frank	6.646	11.567	6.624	0.266	-17.757
Gaussian	6.965	11.305	7.159	0.249	-0.922
Rotated BB6	6.936	11.120	8.405	0.215	-2.905, -2.353
Rotated Gumbel 90	7.191	11.427	6.432	0.276	-3.300

Our results shows that as negative dependency increases, the number of hard failures increases, as expected and like the previous example, the survival function for the independent case declines more rapidly than the survival functions of different copula functions for the negative dependency case. As expected, when the data are more dependent, the distance between the survival function curves increases.

4 Conclusion

The approach given in this paper shows that analysis of dependency is very important, particularly when the degradation process and traumatic failure time are dependent in a general form and have

two different structures (distributions). Our research has indicated that the estimation of survival function using different copulas is different from independent case. This method provides a suitable approach to dependency analysis in complicated cases.

References

[1] Dupuy, J. F. (2014), Accelerated Failure Time Models: A Review, *International Journal of Performability Engineering*, **10(1)**, 23-29. Rf. HAL: hal-01070593

[2] Haghighi, F. and Nazeri Rad, N. (2011), Parametric Degradation Model with Multiple Competing Risks, *Quality Technology & Quantitative Management*, **8(4)**, 389-400.

[3] Haghighi, F. and Nikulin, M. (2010), On the linear degradation model with multiple failure modes, *Journal of Applied Statistics*, **37(9)**, 1499-1507.

[4] Meeker, W. Q. and Escobar, L. A. (1998), *Statistical Methods for Reliability Data*, New Jersey: John Wiley & Sons, Inc.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Analysis of Masked Data with non-ignorable missing mechanism

Misaii, H. [1] and Eftekhari, S. [2]

[1,2] Department of Statistics, Faculty of Mathematics and Statistics, University of Tehran, Tehran, Iran

Abstract

In this article we consider series system products with independent lifetimes for each component in which causes of system failures might be masked Due to cost considerations or environmental restrictions. As a new approach we have defined a missing indicator for the true cause of failure which might be observed or masked. The likelihood with missing not at random (MNAR) mechanism is derived and compared with the usual likelihood for masked data via some simulation studies. The results show superior performance of our approach when non-ignorable missing mechanism is occurred.

Keywords: Masked Data , Non-ignorable missing data, Reliability Analysis.

1 Introduction

In general, failure time data for a series system contain the time to failure along with information on the exact component responsible for the system failure. Which can be used to estimate system and component reliabilities. In many cases, however, due to lack of proper diagnostic equipment or cost and time constraints, the exact component causing the system failure is not identified, but the cause of failure is only narrowed down to a smaller set of components. These Kinds of data are called to be masked (Miyakawa;1984, Basu, et al;1999). In the analysis of masked data, often it is assumed that the masking probability is independent of the component which caused the failure or in the case of dependency some prior probabilities are assumed for these conditional events. Also masking probability is assumed to be dependent on the observed failure time. In this article, as a new approach we will introduce a missing indicator according to the masking status of each

[1]hasanmisaii14@gmail.com
[2]s.eftekhari@khayam.ut.ac.ir

observed failure time. Also a MNAR mechanism (Little Rubin 2002) is assumed for this variable which allow masking to be dependent on the component index along with the observed failure time using generalized linear model. We have performed some simulation studies under exponentially distributed failure times which show superior results for our proposed model when masking has a non-ignorable mechanism.

2 Assumption and Likelihood function

Consider series systems with J components. Assume the observed failure data are $t_1, t_2, ..., t_r$,(n-r systems are censored) but the exact associated cause of failure might be unknown, which is only known to belong to the Minimum Random Subset (MRS) of $\{1, 2, ..., J\}$. Let M_i be the observed MRS corresponding to the failure time $t_i, i = 1, 2, ..., r$. The set M_i essentially consists of components, narrowed down to be the possible causes responsible for the system failure. When M_i is a singleton set, then the corresponding system has an observed cause of failure. While if $M_i = \{1, ..., J\}$ the system is called to be completely masked. We define the binary variable R_i which takes the value 1, when M_i is a singleton and has a zero value for masked data(when M_i has more than one element). Thus, the observed data are

$$(t_1, M_1, R_1), (t_2, M_2, R_2), ..., (t_r, M_r, R_r) \tag{2.1}$$

The model considered in this article is based on the following assumptions.

- The failure of a system occurs due to one of the J independent components with lifetimes, $T_1, T_2, ..., T_J$, and the failure time of the system (T) is the shortest of $T_l, l = 1, ..., J$.
- T_l, the failure time of the lth component , follows a continuous distribution with density and reliability functions denoted by $f_l(t), R_l(t)$.
- $Pr(M = M_i|T = t_i, K_i = l)$ is called the masking probability, where K_i denotes the index of the component actually causing the ith system to fail. $p_l(M_i)$s have some constraints. Let M_i be the collection of all 2^{J-1} possible nonempty subsets of $\{1, ..., J\}$ and $M_l = \{M_0 \in M : l \in M_0, l \in \{1, ..., J\}\}$ and $p_l(M_i) = P(M = M_i|K_i = l) = 0, \forall M_i \in M_l^c = M - M_l$ and thus $\sum_{M_i \in M} p_l(M_i) = \sum_{M_i \in M_l} p_l(M_i) = 1, l = 1, ..., J$. Denote $p_l = \{P_l(M_i) : M_i \in M_l\}$ and $p = (p_1, ..., p_J)$.
- R_i is a Bernoulli variable with success probability $p(R_i = 1|k_i = j, M_i, t_i; j \in M_i) = h(\beta_0 + \beta_1 k_i + \beta_2 t_i)$, where h(.) is some appropriate link function (e.g. logit, probit, clog-log,...). When $\beta_1 = 0$ the missing is ignorable.

Let τ be the end time of the test and $I_{mask} = \{1 \leq i \leqslant r; R_i = 0\}$ denotes the set of indices for maked systems. Therefor the complete likelihood function for data 2.1 is as follows:

$$\begin{aligned} L(\theta) &= \prod_{i \in I_{mask}^c} [p(R_i = 1|k_i = j, t_i, M_i = \{j\}) p(M_i = \{j\}|k_i = j, t_i) \\ &\times f_j(t_i) p(k_i = j)] \prod_{i \in I_{mask}} [p(R_i = 0|k_i = j, t_i, M_i) \\ &\times p(M_i|k_i = j, t_i) f_j(t_i) p(k_i = j) [\prod_{l=1}^{J} R_l(\tau)]^{n-r} \end{aligned} \tag{2.2}$$

If the missing mechanism is at random ($\beta_1 = 0$) the above likelihood reduces to

$$L(\theta) \propto \prod_{i=1}^{r} \left[f(R_i|t_i) \sum_{j \in M_i} f(t_i|k_i, t_i) p(M_i|k_i, t_i) p(k_i = j) \right] \left[\prod_{l=1}^{J} R_l(\tau) \right]^{n-r}$$

Where R_is could be ignored and simple masked data analysis be used.

3 Simulation Study

In this section we assume 100 series systems with two components where the lifetimes of components follow some exponential distribution with parameters α_1 and α_2 respectively for first and second component. We have generated non-ignorable missing mechanism according to the logistic regression $logit(p(R_i = 1|k_i = j)) = \beta_0 + \beta_1 k_i$. The results of maximum likelihood estimation of the parameters α_1 and α_2 are presented in Table 1. In this table the true value of parameters along with the percent of failure due to second component ,CP, and masking rate,MP, are given. Also

Table 1: The results of simulation analysis

	α_1	α_2	β_1	β_0	CP	MP	Bα_1	Bα_2
MAR	0.3	0.7	2.5	-2	70	47	0.062	0.059
MNAR							0.033	0.036
MAR	0.3	0.7	2.5	-3	70	27	0.04	0.024
MNAR							0.011	0.024
MAR	0.3	0.7	2.5	-4	70	13	0.011	0.02
MNAR							0.012	0.002
MAR	0.3	0.7	3.5	-3.5	69	35	0.081	0.073
MNAR							0.002	0.006
MAR	0.3	0.7	3.5	-4	70	26	0.041	0.044
MNAR							0.007	0.005
MAR	0.3	0.7	3.5	-5	70	13	0.02	0.009
MNAR							0.002	0.008
MAR	0.3	0.5	2.5	-4	62	12	0.034	0.018
MNAR							0.007	0.009
MAR	0.3	0.5	2.5	-3	63	25	0.041	0.043
MNAR							0.021	0.019
MAR	0.3	0.5	2.5	-2	63	44	0.082	0.076
MNAR							0.041	0.047
MAR	0.3	0.5	3.5	-5	61	12	0.034	0.021
MNAR							0.005	0.007
MAR	0.3	0.5	3.5	-4	63	25	0.053	0.037
MNAR							0.009	0.025
MAR	0.3	0.5	3.5	-3	62	41	0.117	0.108
MNAR							0.007	0.016

the last two columns present the amount of biasness for α_1 and α_2 (respectively denoted by $B\alpha_1$ and $B\alpha_2$) in 100 iterations of each simulation study. According to the results, MNAR modeling leads to less biased estimators compared with the usuall MAR model.

References

[1] Basu, S. Basu, A. P. and Mukhopadhyay, C. (1999). Bayesian analysis for masked system failure data using nonidentical weibull models, *J.Statist. Plann. Inference*, **78**, 255275.

[2] Little, R. J. A., Rubin, D. B. (2002). Statistical Analysis With Missing Data, 2nd Edition, *Joh Wiley Sons: New York*.

[3] Miyakawa, M. (1984), Analysis of incomplete data in a competing risks model, *IEEE Transactions on Reliability*, **33**, 293-296.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

An algorithm to transform the generalized joint signatures

Mohammadi, L. [1] and Zarezadeh, S. [2]

[1,2] Department of Statistics, Shiraz University, Shiraz 71454, Iran

Abstract

The generalized joint signature (GJS) is a useful tool to investigate the reliability of a set of systems with some shared components. The GJS depends only on the system structure and is a generalization of the signature which is used as an important concept for studying the system reliability. This paper surveys the relation between the GJS, the signature and the GJS of fewer number of systems. An algorithm is provided to obtain the signature and GJS of lower orders from the GJS.

Keywords: Reliability, coherent system, order statistics.

1 Introduction

In the last decades, the modeling and analysis of systems reliability have been widely studied under different approaches. One of the approaches, introduced in [4], is based on the assumption that a system with lifetime T consists of n components having independent and identically distributed (i.i.d.) lifetimes $X_1, X_2, \ldots, X_n$. Under this setting, the system reliability can be presented as $P(T > t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t)$, where $s_i = P(T = X_{i:n})$, $i = 1, \ldots, n$, and $X_{i:n}$ is the i-th ordered component lifetime. The vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ is called the *signature* of the system and depends only on the system structure. Many signature based properties of system reliability are examined by others.

An example of systems with shared components is in computer sciences. For instance, some computer systems include a set of slave computers which share a common server. The reliability of two systems which share some components has been explored in [3]. Consider m systems, with

[1]mohammadileila@yahoo.com
[2]s.zarezadeh@shirazu.ac.ir

lifetimes $T_1, \ldots, T_m$, having some shared components. Let A be a set of components with i.i.d. lifetimes $X_1, \ldots, X_n$ having common distribution function F. Assume that the components of each system is a subset of A such that the union of the components supporting all systems is equal to the set A. If $(i_1, i_2, \ldots, i_m)$ is a permutation of $(1, 2, \ldots, m)$, then the joint distribution function of $(T_1, \ldots, T_m)$, for $t_{i_1} \leq \cdots \leq t_{i_m}$, is

$$G(t_1, t_2, \ldots, t_m) = \sum_{l_1=1}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_m=0}^{n} s_{l_1,l_2,\ldots,l_m}^{(i_1,i_2,\ldots,i_m)} \prod_{j=1}^{m} F_{l_j:n}(t_{i_j}), \tag{1.1}$$

where $F_{l_j:n}$ is the distribution function of the l_j-th order statistic from $X_1, ..., X_n$ and $s_{l_1,l_2,\ldots,l_m}^{(i_1,i_2,\ldots,i_m)}$'s are coefficients, such that $\sum_{l_1,\ldots,l_m} s_{l_1,\ldots,l_m}^{(i_1,\ldots,i_m)} = 1$. Let $S^{(i_1,\ldots,i_m)}$ be an m-dimensional matrix defined as below $S^{(i_1,\ldots,i_m)} = \{s_{l_1,\ldots,l_m}^{(i_1,\ldots,i_m)}\}_{l_1 \in \{1,\ldots,n\},\ l_2,\ldots,l_m \in \{0,\ldots,n\}}$. The vector $(S^{(i_1,\ldots,i_m)})_{(i_1,\ldots,i_m)}$, with $m!$ elements, is referred to as the generalized joint signature (GJS) of the systems. Each $S^{(i_1,\ldots,i_m)}$ is called the GJS for the corresponding permutation $(i_1, \ldots, i_m)$. Notice that the elements $s_{l_1,\ldots,l_m}^{(i_1,\ldots,i_m)}$ are not necessarily non-negative, although they sum up to 1.

In Section 2, via an example, a general formula for transforming the GJS to JS and then to signature is given. In Section 3, we propose an algorithm for the transformation of the GJS's of different orders. Via an R program, we generate the results in an example. It will be implemented in an R package later.

2 The relation between the GJS, JS and signature

Assume that we have the GJS of m systems. We want to know how one can obtain the GJS of k systems, $k < m$. Let us explore this relationship in the following example.

Example 1. Consider three systems with lifetimes $T_1 = \max\{X_1, X_2\}$, $T_2 = \max\{X_2, X_3\}$, and $T_3 = \max\{X_3, X_4\}$. Suppose X_1, X_2, X_3, X_4 are i.i.d. random variables with common distribution function F. The GJS of (T_1, T_2, T_3) is obtained in Example 3.5 in [2]. Now we want to calculate the JS of the two systems (T_1, T_2) from the GJS of (T_1, T_2, T_3). To obtain the joint distribution function of (T_1, T_2), we need to assume $t_3 \to \infty$, hence, we only consider the permutations in which $\max\{t_1, t_2, t_3\} = t_3$. Let us assume $t_1 \leq t_2 \leq t_3$. The GJS matrix $S_1^{(1,2,3)}$ is a 5×5 zero matrix and $S_2^{(1,2,3)} = \frac{1}{96}Q$, $S_3^{(1,2,3)} = \frac{1}{48}Q$, $S_4^{(1,2,3)} = \frac{1}{32}Q$, where Q is a 5×5 matrix with the first row and column all 0's and the rest all 1's. We make the sum of the GJS matrices over the third dimension. Hence $S_1^{(1,2,3)}$ is now a zero vector of length 5 and $S_2^{(1,2,3)} = \frac{1}{24}\mathbf{q}$, $S_3^{(1,2,3)} = \frac{1}{12}\mathbf{q}$, $S_4^{(1,2,3)} = \frac{1}{8}\mathbf{q}$, where $\mathbf{q} = (0, 1, 1, 1, 1)$. Arranging all the vectors as the rows of a matrix, one obtains $S^{(1,2)}$ as a 4 by 5 matrix with the first column and row all 0's. The other elements of the second, third and fourth rows are $\frac{1}{24}$, $\frac{1}{12}$ and $\frac{1}{8}$, respectively. This is the JS for (T_1, T_2), for the permutation $(1, 2)$, when we consider four components. In order to obtain the JS for (T_1, T_2), with the three components X_1, X_2, X_3, we delete the last row and column of $S^{(1,2)}$. The elements of the matrix obtained do not sum up to 1. We normalize the matrix and obtain a matrix which is exactly the S matrix in Example 2.1 of [3], for the same (T_1, T_2). Choosing the permutation $t_2 \leq t_1 \leq t_3$, a similar procedure leads us to $S^{(2,1)} = (S^{(1,2)})^T$, which is obtained as S^* in Example 2.1 of [3]. Repeating the same argument as above with the last $S^{(1,2)}$, one derives $\mathbf{s} = (0, \frac{1}{3}, \frac{2}{3})$, which is for the case of three components. If we want to obtain the $\mathbf{s}$ for the case of two components, X_1, X_2, we delete the last element and normalize the vector and derive $\mathbf{s} = (0, 1)$.

3 The proposed algorithm

The notations used in the proposed algorithm are as below.
Notations:

$Perm(i)$	a permutation $(i_1, \ldots, i_m)$ of $(1, ..., m)$.
S	the vector of GJS matrices of $T_1, ..., T_m$.
$S^{(i_1,...,i_m)}$	the GJS matrix of $T_1, ..., T_m$ for $(i_1, \ldots, i_m)$.
S_j	the vector $(S^{(i_1,...,i_m)})_{(i_1,...,i_m):\{j_1,...,j_k\}=\{i_1,...,i_k\}}$.
N_j	the number of elements of S_j.
$S_{j,l}^{(i_1,...,i_m)}$	each member of S_j, for $l = 1, ..., N_j$.
$S_{j,l}^{(i_1,...,i_k)}$	the GJS matrix of $(T_{j_1}, ..., T_{j_k})$, corresponding to $S_{j,l}^{(i_1,...,i_m)}$, with the original n components, for $l = 1, ..., N_j$.
n_1	the maximum number of components used in $(T_{j_1}, ..., T_{j_k})$.
$S_{j,l,[n_1]}^{(i_1,...,i_k)}$	the non-normalized GJS matrix of $(T_{j_1}, ..., T_{j_k})$, related to the n_1 components, for $l = 1, ..., N_j$.
$S_{j,[n_1]}^{(i_1,...,i_k)}$	the vector $(S_{j,l,[n_1]}^{(i_1,...,i_k)})_{l=1,...,N_j}$.
$[S_{j,l,[n_1]}^{(i_1,...,i_k)}]$	the total mass of $S_{j,l,[n_1]}^{(i_1,...,i_k)}$.
$S_{j,l,[n_1],1}^{(i_1,...,i_k)}$	the normalized $S_{j,l,[n_1]}^{(i_1,...,i_k)}$.
$S_{j,[n_1],1}^{(i_1,...,i_k)}$	the vector $(S_{j,l,[n_1],1}^{(i_1,...,i_k)})_{l=1,...,N_j}$.

The Algorithm

1. Insert S, Perm(i) and $j = (j_1, ..., j_k)$.
2. Create S_j.
3. Create $S_{j,l}^{(i_1,...,i_m)}$, for $l = 1, ..., N_j$.
4. Create $S_{j,l,[n_1]}^{(i_1,...,i_m)}$, for $l = 1, ..., N_j$.
5. Create $S_{j,[n_1]}^{(i_1,...,i_k)}$.
6. Create and print $S_{j,[n_1],1}^{(i_1,...,i_k)}$.

Here is an example for the algorithm.

Example 2. Let X_1, X_2, X_3, X_4 be i.i.d. random variables with common distribution function F. Consider now four systems with lifetimes $T_1 = \min\{\max\{X_1, X_2\}, X_3\}$, $T_2 = X_3$, $T_3 = \max\{X_1, X_4\}$, and $T_4 = \max\{X_2, X_4\}$. The GJS of (T_1, T_2, T_3, T_4) is already obtained in Example 3.4 in [2] and now, we want to obtain the JS for T_1 and T_2. We apply Algorithm 1 and the output for all permutations starting with (1,2) is shown in Listing 1.

Listing 1

Per

[1] 1 2

S

	$F_{0:4}(t2)$	$F_{1:4}(t2)$	$F_{2:4}(t2)$	$F_{3:4}(t2)$	$F_{4:4}(t2)$
$F_{1:4}(t1)$	$1/4$	0	0	0	0
$F_{2:4}(t1)$	$1/4$	$1/24$	$1/24$	$1/24$	$1/24$
$F_{3:4}(t1)$	0	$1/12$	$1/12$	$1/12$	$1/12$
$F_{4:4}(t1)$	$-1/2$	$1/8$	$1/8$	$1/8$	$1/8$

References

[1] Gertsbakh, I. and Shpungin, Y. (2009), *Models of Network Reliability. Analysis, Combinatorics, and Monte Carlo*, CRC Press.

[2] Mohammadi, L., and Zarezadeh, S., (2016). Algorithms to Access the Generalized Joint Signature of Systems with Shared Components. *Submitted.*

[3] Navarro, J., Samaniego, F. J. and Balakrishnan, N. (2010). The joint signature of coherent systems with shared components, *Journal of Applied Probability* **47**(1), 235-253.

[4] Samaniego, F.J. (1985), On closure of the IFR class under formation of coherent systems, *IEEE Trans. Reliab.*, **34**, 69-72, 1985.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Optimal Burn-in Test for Bivariate Degradation Model

Mosayebi, E. [1] Shemehsavar, S. [2]

[1,2] School of Mathematics, Statistics and Computer Science, Faculty of Science, University of Tehran, 14155-6455, Tehran, Iran

Abstract

Burn-in test is very important in manufacturing process which helps to detect weak items before selling products. Traditional burn-in test which was based on collecting time to failure data is not efficient in short time. When there is at least one quality characteristic (QC) related to lifetime of the products, collecting degradation data and using them in inference can be helpful. New products, due to their complexity, have more than one QC. these QCs may be dependent and so it is very important to find the joint distribution of them. In this paper, we assume that the product has two QCs and their degradation can be governed by inverse Gaussian process. The optimal burn-in test using cost model is obtained and finally a simulation study is conducted.

Keywords: Burn-in test, Bivariate degradation, optimal test, Stochastic process, Inverse Gaussian process

1 Introduction

To satisfy their customers, manufacturers try hard to produce high reliable products. Uunfortunately due to some flaws in the production procedure and the quality of used material, usually a subpopulation of products can not operate acceptably. These products which are called weak items fail sooner than typical items. If they are not removed before selling products, it will impose burden of finances. Burn-in test helps manufacturers to detect and eliminate weak items.

Traditionally, burn-in test was based on collecting failure data. Nnowadays, products function for a long time and then their failure time cannot be obtained, so the traditional burn-in

[1]elhammosayebi2@ut.ac.ir
[2]Shemehsavar@khayam.ut.ac.ir

test can be efficient no longer. In such cases, using degradation data related to some quality characteristic (QC) of the product can solve the problem. Due to the nature of the degradation data and their dependency on time, using stochastic processes is an appropriate way to model degradation process. Important processes used for this issue are Wiener process, gamma process, and inverse Gaussian process.

Tseng and Tang [7],Tseng et al. [8], Tseng and Peng [6], and Ye et al.[9]developed different decision rules for distinguishing weak items with considering degradation follows a Wiener process. Burn-in test based on gamma process is discussed in Tsai et al. [5] and Ye et al.[10]. Also the optimal burn-in test for inverse Gaussian process is worked out by Zhang et al [19] .

Nowadays, the products are more complex and it means that they may have multiple degradation processes. Usually these multiple degradation processes depend each other and so their dependency must be considered in the model. There are two approaches for this issue: using a multivariate process and using copula. In the former approach, the relation between multiple degradation process are govern by a multivariate process; Jin and Matthews [2], Pham and Mercier [3], to name a few. Works related to former approach can be found in: Hao and Su [1], Pan and Sun[4]. In this paper, we are to discuss obtaining optimal burn-in test in presence of dependent bivariate degradation processes. We consider the bivariate degradation follows a bivariate inverse Gaussian process. The remainder of this paper is organized as follows. Section 2 gives some basic information of model and discusses optimal burn in test and the optimal termination time. Section 3 deals with statistical inference and parameters estimation.

2 Model Description

Let $(X_1(t), X_2(t)), t > 0$ be the bivariate degradation path of the quality characteristics of a product. Assume that the bivariate degradation follows a bivariate process, that is $(X_1(t), X_2(t))$ has a distribution $f_\Theta(x_1, x_2)$. Usually, products can be classified into two groups. Weak group with higher tendency to fail and typical group. Tacking this into account, we have:

$$(X_1(t), X_2(t)) \sim \begin{cases} f_{\Theta_w}(x_1, x_2) & \text{for weak group} \\ f_{\Theta_t}(x_1, x_2) & \text{for typical group} \end{cases} \tag{2.1}$$

such that $E_{\Theta_w}(X_i(t)) > E_{\Theta_t}(X_i(t))$, for $i = 1, 2$. Hence, from 2.1, an item is classified as a typical one if $X_1(t)$ and $X_2(t)$ are relatively small. Let $\xi_1(t)$ and $\xi_2(t)$ denote cutoff points for $X_1(t)$ and $X_2(t)$ between the typical and weak items, respectively. Then the decision rule is given by:

$$\text{An item is classified as a weak item} \Leftrightarrow X_1(t) < \xi_1(t) \text{ and } X_2 t) < \xi_2(t).$$

So for the fixed time t, the probabilities of type I and type II errors are:

$$\begin{aligned} \alpha(t) &= 1 - P(X_1(t) < \xi_1(t), X_2(t) < \xi_2(t) | \text{typical item}) \\ &= 1 - \int_0^{\xi_1(t)} \int_0^{\xi_2(t)} f_{\Theta_t}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

$$\begin{aligned} \beta(t) &= P(X_1(t) < \xi_1(t), X_2(t) < \xi_2(t) | \text{weak item}) \\ &= \int_0^{\xi_1(t)} \int_0^{\xi_2(t)} f_{\Theta_w}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

Each of these errors impose some cost to model. Let C_α and C_β be the cost of type I and type II errors respectively. Then the misclassification cost, sum of the cost of both errors, at time t is :

$$MC(t) = C_\alpha n(1-p)\alpha(t) + C_\beta np\beta(t), \tag{2.2}$$

where n is the total number of items subjected to burn-in test and p is the proportion of weak items. For a fixed t, the optimal $(\xi_1^*(t), \xi_2^*(t))$ is given by

$$(\xi_1^*(t), \xi_2^*(t)) = \arg\min_{(\xi_1(t),\xi_2(t))} MC(t) \tag{2.3}$$

In addition to the misclassification cost, there is another burden of cost. Operation cost (OC), the cost of conducting a degradation test and collecting data, must be taken into account. The total cost (TC) is the sum of these two costs; i.e.

$$TC(t_i) = MC(\xi(t_i), t) + C_{op} n t_i + C_{mea} n(i+1) \tag{2.4}$$

Similarly, the optimal termination time can be determined by:

$$t_i^* = \arg\min_{\{t_i\}_{i=1}^m} \{MC(\xi^*(t_i), t_i) + C_{op} n t_i + C_{mea} n(i+1)\} \tag{2.5}$$

where C_{op} and C_{mea} are the cost of running a degradation test and cost of measuring degradation of an item, respectively.

3 Statistical Inference

From 2.1, the distribution of $(X_1(t), X_2(t))$ is a mixture distributions. That is

$$f(x_1, x_2) = p f_{\Theta_w}(x_1, x_2) + (1-p) f_{\Theta_t}(x_1, x_2).$$

Assume that n and m are the total number of products under burn-in test and the number of inspection, respectively. We define:

$$Y_k^{(ij)} = X_k^i(t_j) - X_k^i(t_{j-1}), \qquad k = 1, 2$$

where $X_k^i(t_j)$ denotes the degradation of k-th QC measured for i-th item at time t_j. Then the log-likelihood has the following form:

$$l(\Theta) = \sum_{i=1}^{n} \ln\{p H_i(\Theta_w) + (1-p) H_i(\Theta_t)\} \tag{3.6}$$

where

$$H_i(\Theta) = \prod_{j=1}^{m} f_\Theta(y_1^{(ij)}, y_2^{(ij)})$$

The maximum likelihood estimators(MLEs) of unknown parameters can be obtained by maximizing 3.6.

References

[1] Hao H, Su C. Bivariate Nonlinear Diffusion Degradation Process Modeling via Copula and MCMC. *Mathematical Problems in Engineering* 2014; ID 510929.

[2] Jin G, Matthews D. Measurement Plan Optimization for Degradation Test Design based on the Bivariate Wiener Process. *Qual. Reliab. Engng. Int.* 2014; 30: 1215-1231.

[3] Pham H.H, Mercier S. An imperfect replacement policy for a periodically tested system with two dependent wear indicators.ESREL 2013 (Annual European Safety and Reliability Conference), Amsterdam,Oct. 2013.

[4] Pan Z, Sun Q. Optimal Design for Step-Stress Accelerated Degradation Test with Multiple Performance Characteristics Based on Gamma Processes. *Communications in Statistics-Simulation and Computation* 2014 , 43: 298-314.

[5] Tsai C.C., Tseng S.T. and Balakrishnan N. (2011). Optimal burn-in policy for highly reliable products using gamma degradation process. IEEE Transactions on Reliability 60(1): 234-245.

[6] Tseng S. T. and PengC. Y.(2004). Optimal burn-in policy by using an integrated Wiener process. IIE Transactions 36: 1161-1170.

[7] Tseng S. T. and Tang J. (2001). Optimal burn-in time for highly reliable products. International Journal of Industrial Engineering 8: 329-338.

[8] Tseng S.T., Tang J. and Ku L.H.(2003). Determination of burn-in parameters and residual life for highly reliable products. Naval Research Logistics 50:1-14.

[9] Ye Z. S. , Shen Y. and Xie M.(2012). Degradation-based burn-in with preventive maintenance. European Journal of Operational Research 221:360-367.

[10] Ye Z. S. , Xie M. ,Tang L.C. and Shen Y. (2012). Degradation-based burn-in planning under competing risks. Technometrics 54(2): 159-168.

[11] Zhang M., Ye Z. and Xie M.(2014).Optimal burn-in policy for highly reliable products using inverse Gaussian degradation process. Engineering Asset Management - Systems, Professional Practices and Certification, Lecture Notes in Mechanical Engineering, DOI 10.1007/978-3-319-09507-3_ 86.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Concomitant of order statistics arising from non-identical independent random vector

Rezapour, M. [1] Sheikhi, A. [2] Negarestani, H. [3]

[1,2] Shahid Bahonar University of Kerman, Department of Statistics, Faculty of Mathematics and Computer, Kerman, Iran.
[3] Department of Statistics, Faculty of Mathematical Sciences,Vali-e-Asr University of Rafsanjan, Iran.

Abstract

In this paper we derive the pdf and cdf of the concomitant of the r-th order statistics arising from a dependent and non-identical random vector $(X_i, Y_i), i = 1, \ldots, n$ and then present their formulas in a special case when X_i and Y_j are independent for $i \neq j = 1, \ldots, n$. We also derive some formulas for the pdf and cdf of concomitant of order statistics arising from a random vector distributed according to an (nested) Archimedean copula. Some explicit formulas for the pdf of concomitant of order statistics was obtained for Clayton and Gumbel families. A numerical example provided for more illustration.

Keywords: Archimedean copula, order statistics, reliability systems.

1 Introduction

Consider a random sample $(X_i, Y_i), i = 1, \ldots, n$, from a bivariate cdf. If the pairs are ordered by their X variates, then the Y variate associated with the r-th order statistic $X_{r:n}$ of X will be denoted by $Y_{[r:n]}, 1 \leq r \leq n$, and called the concomitant of the r-th order statistic. Concomitants of order statistics have found a wide variety of applications in different fields. The most important use of concomitants arises in selection procedures when $k(1 \leq k \leq n)$ individuals are

[1]mohsenrzp@uk.ac.ir
[2]sheikhy.a@uk.ac.ir
[3]H_Negarestani@yahoo.com

chosen on the basis of their X values. Then the corresponding Y values represent performance on an associated characteristic. For example, X might be the score of a candidate on a screening test and Y the score on a later test. In the literature, there are many papers which deal with concomitants. David, O'Connell and Yang [3] derived the cdf of the rank of $Y_{[r:n]}$. David and Galambos [2] discussed the asymptotic properties of the rank of $Y_{[r:n]}$. Balasubramanian and Beg [1] studied the concomitants for bivariate exponential cdfs of Marshall and Olkin, Morgenstern type bivariate exponential cdfs and Gumbel's bivariate exponential cdf, respectively, and gave the recurrence relations between single moments and product moments of concomitants of order statistics.

2 Concomitants of order statistics arising from non-iid random variables

In this section, we consider the pdf and cdf of the concomitants of the r-th order statistics arising from a non-iid random vectors $(X_i, Y_i), i = 1, \ldots, n$. We denote the pdf and cdf of the concomitant of r-th order statistic $Y_{[r:n]}, 1 \leq r \leq n$, denoted by $g_{[r:n]}(y)$ and $G_{[r:n]}(y)$, respectively.

Theorem 1. *The pdf and cdf of the concomitant of the r-th order statistics arising from a non-iid random vectors $(X_i, Y_i), i = 1, \ldots, n$, respectively, given by*

$$g_{[r:n]}(y) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sum_{C_{r-1}} P(_{C_{r-1}} \leq x, _{C^c_{r-1}} > x | Y = y, X_i = x) f_{Y,X_i}(y, x) dx \,, \tag{2.1}$$

and

$$G_{[r:n]}(y) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sum_{C_{r-1}} P(Y_i \leq y, _{C_{r-1}} \leq x, _{C^c_{r-1}} > x | X_i = x) f_{X_i}(x) dx \,, \tag{2.2}$$

where the summations over C_{r-1} extend to all permutations $(s_1, \ldots, s_{n-1})$ of $\{1, \ldots, i-1, i+1, n\}$ for which $s_1 < \cdots < s_{r-1}$ and $s_r < \cdots < s_n$ and and $_{C_{r-1}} = (X_{s_1}, \ldots, X_{s_{r-1}})$ and $_{C^c_{r-1}} = (X_{s_r}, \ldots, X_{s_{n-1}})$ and denote a vector of ones with desirable dimension and $f_{X_i}(x)$ and $f_{Y,X_i}(y, x)$, $i = 1, \ldots, n$ are the marginal pdfs of the random variable X_i and random vector (Y, X_i), respectively.

In the following result, we consider the pdf and cdf of r-th order statistics arising from a non-iid random vectors $(X_i, Y_i), i = 1, \ldots, n$, with additional assumption in which Y_i is independent of X_j for any $i \neq j = 1, \ldots, n$.

Theorem 2. *Under assumption of Theorem 2, the pdf and cdf of the concomitant of the r-th order statistics, respectively, given by*

$$g_{[r:n]}(y) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} f_{Y_i|X_i}(y|x) \sum_{C_{r-1}} P(_{C_{r-1}} \leq x, _{C^c_{r-1}} > x | X_i = x) f_i(x) dx \,, \tag{2.3}$$

and

$$G_{[r:n]}(y) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} P(Y_i \leq y | X_i = x) \sum_{C_{r-1}} P(_{C_{r-1}} \leq x, _{C^c_{r-1}} > x | X_i = x) f_i(x) dx \,, \tag{2.4}$$

where $f_{Y_i|X_i}(y|x)$ is the conditional pdf of Y_i given X_i, for $i = 1, \ldots, n$.

3 Numerical example

Johnson and Wichern ([4], p.24) provide data consisting of mineral content measurements of three bones (radius, humerus, ulna) in two arms (dominant and non dominant) for each of 25 old women. We consider variables, X as Dominant radius and Y as Non dominant radius. The sample data is presented in Table 1. We apply model to this data. For this purpose we fit a two dimensional Archimedean copula to this data set. We use inference from marginal method. In this method we first fit the best marginal distribution to this data and then we obtain the best Archimedean copula fitted to this data set. The first procedure is done using "easy fit" package. Using this package the Dagum distribution with parameters $k = 0.28751$, $\alpha = 20.237$, and $\beta = 0.88839$ and $\gamma = 0$ is the best distribution fitted to the Dominant radius data, the histogram of this data is shown in Figure 1 and the Bur distribution with parameters $k = 5.6456$, $\alpha = 7.9541$, and $\beta = 0.98628$ and $\gamma = 0$ is the best distribution fitted to the Dominant radius data, the histogram of this data is shown in Figure 2. Then using Copula package in R the Gumbel copula with parameter $\hat{\theta} = 2.9812$ is the best Archimedean copula fitted to the data set.

Table 1. Goodness of fit test of the data set

Copula	P-value	Log-likelihood
Clayton	2.11×10^{-7}	21.97
Frank	1.49×10^{-11}	33.63
Gumbel	$< 2 \times 10^{-16}$	36.49
Ali-Mikhail-Haq	2.55×10^{-14}	33.72

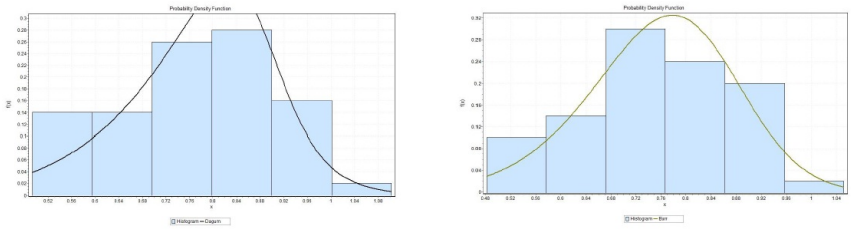


Figure 1: The plots of distribution functions of DAGUM and Burr fitted to the dominant radius and non dominant radius data, respectively.

4 Conclusion

In this paper we derive the pdf and cdf of the concomitant of the r-th order statistics arising from a dependent and non-identical random vector $(X_i, Y_i), i = 1, \ldots, n$ and then present their formulas in a special case when X_i and Y_j are independent for $i \neq j = 1, \ldots, n$. However, there are many interesting further work which may be carry out. We may generalize our work by extending our results in the presence of one or more covariates. The results of this paper may be extended to elliptical copula distributions.

References

[1] K. Balasubramanian, M.I. Beg, Concomitant of order statistics in Gumbels bivariate exponential distribution, Sankhya Ser. B 60 (1998) 399-406.

[2] H.A. David, J. Galambos, The asymptotic theory of concomitants of order statistics, J. Appl. Probab. 11 (1974) 762-770.

[3] H.A. David, M.J. OConnell, S.S. Yang, distribution and expected value of the rank of a concomitant of an order statistic, Ann. Statist. 5 (1977) 216-223.

[4] Johnson, R. A., Wichern, D. W. *Applied Multivariate Statistical Analysis.* 5th Ed., Prentice-Hall, 2002.

[5] A. McNeil, Sampling nested Archimedean copulas, *J. Stat. Comput. Simul.* 78 6 (2008) 567-581.

[6] A. J. McNeil, and J. Nešlehová, Multivariate Archimedean copulas, d-monotone functions and L1-norm symmetric distributions. *Ann. Stat.* 37 (2009) 3059-3097.

[7] A. Sheikhi, Y. Mehrali, M. Tata, On the exact joint distribution of a linear combination of order statistics and their concomitants in an exchangeable multivariate normal distribution, Stat. Paper. 54 (2013) 325332.

[8] A. Sheikhi, A. Jamalizadeh, Regression analysis using order statistics. Stat. Paper. 52 (2011) 885892.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Some design of strength system in stress-strength model

Pakdaman, Z. [1] Ahmadi, J. [2] Doostparast, M. [3]

Department of Statistics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran

Abstract

In this paper, the aim is to design the strength system, so that the stress-strength reliability increase. Stress-strength parameters for several system designs have been introduced and the systems have been compared. The performance of stress-strength reliability parameter in the corresponding system has been assessed.

Keywords: Stress-strength system , Reliability, PHR model

1 Introduction

In a stress-strength model, according to the desired problem, stress-strength reliability parameters can be expressed in different ways, followed by parameter estimation problem arises. In statistical inference, there are many ways to find the desirable estimator for the parameter of interest. In recent years, numerous studies have been carried out on estimation of the stress-strength reliability parameter, under the various distributions on stress-strength with the dependency or in-dependency hypothesis of the components of the model and different plans of sampling scheme. A survey of different topic on the stress-strength model has been covered the following studies Kotz (2003) and Krishnamoorthy et al.(2007). Point and interval estimations for the parameter of stress-strength reliability have been conducted by Al-Mutairi et al. (2015) where lifetime of stress and strength system both are simultaneously distributed as Lindley. Consider a stress-strength system consisting of two strength components with lifetimes W and Y and the stress component with lifetime X.

1

2

3

In this paper the question that has been raised is:" how the two components of the strength system together so that increases the reliability?" Here we would like to review $g(Y,W)$, where $g(.,.)$ is any positive function of two variables Y and W, such as $Y+W$, YW, $\min\{Y,W\}$ and etc.

2 Some design of strength system

Let Y and W denote the lifetimes of strength components with survival function $\bar{G}$ and $\bar{H}$ respectively, and X is the lifetime of the stress system with the survival functions $\bar{F}$. We are interested in computing $R = P(X < g(Y,W))$, where $g(.,.)$ is a given positive function of Y and W. It has been interested to find the effect of the relationship among the strength components on the stress-strength reliability. Let Y and W are independent random variables, then

$$R = P(X < g(Y,W)) = \int_0^\infty \int_0^\infty P(X < g(y,w))dG(y)dH(w). \tag{2.1}$$

In what follows, we consider different designs which lead different forms for functions $g(.,.)$. In each case, we obtain the stress-strength reliability.

•$g(Y,W) = Y + W$
Let the component in strength system fails, Here we replace another component in the strength system. In this case, we have

$$R^* = P(X < Y + W) = \int_0^1 \int_0^1 F(G^{-1}(u_1) + H^{-1}(u_2))du_1du_2. \tag{2.2}$$

Result 1: If $X \leqslant_{pr} Y$ then $R^* \geq \frac{1}{2}$.
Result 2: If $F = \Phi(G)$ or $F = \Phi(H)$ where $\Phi(.)$ is a concave distortion function then $R^* \geq \frac{1}{2}$.
Result 3: If X is NBU and $X \leqslant_{st} Y$ and $X \leqslant_{st} W$ then $R^* \geq \frac{3}{4}$.
Result 4: If X is NBU, for PHR model $R^* \geq 1 - \frac{\alpha\beta}{(\alpha+1)(\beta+1)}$.
Result 5: If X is NBU, for RPHR model $R^* \geq 1 - \frac{1}{(\alpha+1)(\beta+1)}$.

Theorem 1. *If X is NBU then $R^* \geq c + (1-c)a$, where $c = \int_0^1 F(H^{-1}(u))du$ and $a = \int_0^1 F(G^{-1}(u))du$.*

Theorem 2. *Let X is NBU or NWU. X has the exponential distribution if and only if $R^* = a + (1-a)c$, where $c = \int_0^1 F(H^{-1}(u))du$ and $a = \int_0^1 F(G^{-1}(u))du$.*

•$g(Y,W) = \min(Y,W)$
Suppose that the components in the strength system is series . Therefore

$$R_1{}^* = P(X < \min(Y,W)) = \int_0^1 \bar{G}(F^{-1}(u))\bar{H}(F^{-1}(u))du \tag{2.3}$$

Result 6: If $X \leqslant_{st} Y$ and $X \leqslant_{st} W$, then $R_1^* \geq \frac{1}{3}$.

•$g(Y,W) = \max(Y,W)$
Suppose that the components in the strength system, is parallel. Thus

$$R_2{}^* = P(X < \max(Y,W)) = 1 - \int_0^1 G(F^{-1}(u))H(F^{-1}(u))du \tag{2.4}$$

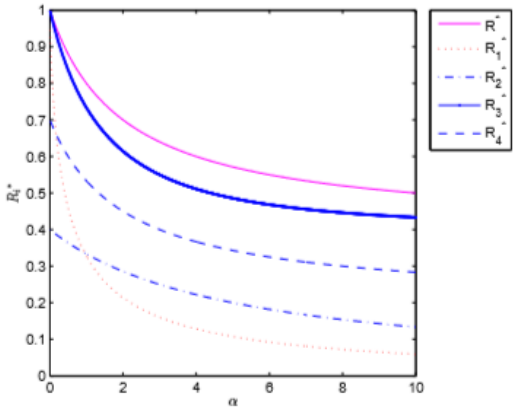


Figure 1: Stress-strength reliability in Table 1 for $\lambda = 2$, $\beta = 3$ and $p_1 = p_2 = \frac{1}{2}$.

Result 7: If $X \leqslant_{st} Y$ and $X \leqslant_{st} W$, then $R_2^* \geq \frac{2}{3}$.
$\bullet g(Y, W) = YW$
we have

$$R_3{}^* = P(X < YW) = \int_0^1 \int_0^1 F(G^{-1}(u_1)H^{-1}(u_2))du_1du_2. \tag{2.5}$$

$\bullet g(Y, W) = V$
Let $g(Y, W) = V$ has the survival function $\bar{F}_V(v) = p_1\bar{G}(v) + p_2\bar{H}(v)$, where $p_1 = P(V = Y)$ and $p_2 = P(V = W)$. By definition

$$R_4^* = \int_0^\infty \bar{F}_V(v)f(v)dv = p_1 \int_0^1 \bar{G}(F^{-1}(u))du + p_2 \int_0^1 \bar{H}(F^{-1}(u))du.$$

Result 8: If $X \leqslant_{st} Y$ and $X \leqslant_{st} W$, then $R_4^* \geq \frac{1}{2}$.

$\bullet g(Y, W) = \min(Y, t) + W$
In this case

$$\begin{aligned} R_5^*(t) &= P(X < \min(Y, t) + W) \\ &= \int_0^\infty \int_0^{G(t)} F(G^{-1}(u_1) + H^{-1}(u_2))du_1du_2 + \bar{G}(t) \int_0^1 F(l + H^{-1}(u_2))du_2. \end{aligned}$$

In the following, for the exponential case, we compare the stress-strength reliability in various position in Figure 1.
Example: Let X, Y and W have exponential distributions with means λ, β and α respectively. Here for different functions of $g(Y, W)$, we obtain the stress-strength reliability. Under this assumptions we have

In Figure 1, $\alpha > 1$, ${R_1}^* < {R_2}^* < {R_4}^* < {R_3}^* < R^*$ is satisfied. This inequality shows that when the strength component has been designed to form $g(Y, W) = Y + W$, we obtain the highest stress-strength reliability among considered designs.

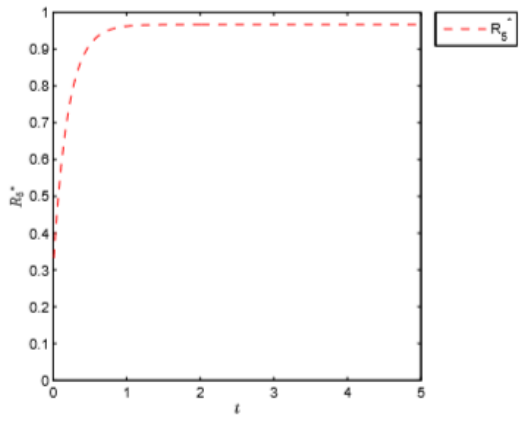


Figure 2: $R_5^*(t)$ for $\lambda = 2$, $\beta = 3$ and $\alpha = 4$

Table 1:

$R^* = 1 - \frac{\alpha\beta}{(\lambda+\beta)(\lambda+\alpha)}$	${R_1}^* = 1 - \alpha\beta \int_0^\infty \frac{e^{-\alpha w}}{\beta+\lambda w}$
${R_2}^* = \frac{\lambda}{\alpha+\beta+\lambda}$	${R_3}^* = \frac{\lambda}{\alpha+\lambda} + \frac{\lambda}{\beta+\lambda} - \frac{\lambda}{\alpha+\beta+\lambda}$
${R_4}^* = p_1 \frac{\lambda}{\lambda+\beta} + p_2 \frac{\lambda}{\lambda+\alpha}$	${R_5}^* = 1 - \frac{1}{(\alpha+\lambda)(\beta+\lambda)} - \frac{\alpha}{\alpha+\lambda} e^{-t(\beta+\lambda)}$

References

[1] Al-Mutari, D. K., Ghitany, M. E. and Kundu, D. (2015) Inferences on stress-strength reliability from Lindley distribution, *Communication in Statistics- Theory and Methods*,**42**, 1443–1463.

[2] Kotz, S., Lumelskii, Y., Pensky, M. (2003) The Stress Strength Model and its Generalization. *World Scientific*, Singapor.

[3] Krishnamoorthy, K., Mukherjee, S., Guo, H. (2007) Inference on reliability in twoparameter exponential stress-strength model. *Metrica*, **65**, 261-273.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Parameters Estimation for the Generalized Inverted Exponential Distribution Based on Progressive Type II Hybrid Censored Sample

Panahi, H. [1]

[1] Department of Statistics, Lahijan Branch, Islamic Azad University, P. O. Box 1616, Lahijan, Iran

Abstract

In this article, we consider the statistical inferences of the unknown parameters of a generalized inverted exponential (GIE) distribution based on the progressive Type II hybrid censored sample. The maximum likelihood estimators are developed for estimating the unknown parameters. The Bayesian estimates of the unknown parameters under the assumption of independent gamma priors are obtained using Lindley's approximation. Finally, one data set is analyzed for illustrative purposes.

Keywords: Generalized Inverted Exponential; Lindley's Approximation; Progressive Type II Hybrid Censoring.

1 Introduction

In many experimental studies, the experimenters may not always be in a position to obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. The most common censoring schemes are Type I (time) and Type II (failure) censoring. A hybrid censoring scheme is the mixture of the Type I and Type II censoring schemes. However, the conventional Type I, Type II and hybrid censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. One censoring scheme known as Type II progressive censoring scheme, which has this advantage. The drawback of the Type II progressive censoring, similar to the conventional Type II censoring, is that it can take a lot of time to get to the m^{th} failure time.

[1] panahi@liau.ac.ir

Because of that problem, Kundu and Joarder (2006) introduced a new censoring scheme named as progressive Type II hybrid censoring (PHCII) scheme, which ensures that the length of the experiment cannot exceed a pre-specified time point T. Although several articles have been done on the progressive Type II hybrid censoring scheme (See for example, Bayat Mokhtari et. al., (2011)), but we have not come across any article under this censoring for the GIE distribution. Thus, in this paper we consider the analysis of the progressive Type II hybrid censored lifetime data when the lifetime of each experimental unit follows a GIE distribution. The probability density function of the GIE distribution is given by

$$f(x;\alpha,\beta) = \frac{\alpha\beta}{x^2}\exp(-\beta/x)[1-\exp(-\beta/x)]^{\alpha-1};\ \alpha>0,\beta>0 \qquad (1)$$

Due to its practicality, it can be used for many applications, including accelerated life testing, horse racing, supermarket queues, sea currents, wind speeds, and others. In Section 2, MLEs of the unknown parameters are obtained. Bayesian analyses are presented in Section 3. Analysis of real data set appears in Section 4.

2 Maximum Likelihood Estimation

Let $X_{1:m,n},...,X_{m:m,n}$ denote the progressively Type II hybrid order statistics from a $GIE(\alpha,\beta)$ distribution. To simplify the notation, we will use X_i in place of $X_{i:m,n}$. Then, based on progressive Type II hybrid censored data, the log-likelihood function from $GIE(\alpha,\beta)$, dropping terms that do not involve α and β is

$$L(\alpha,\beta) = s\ln\alpha + s\ln\beta - 2\sum_{i=1}^{s}\ln x_i - \beta\sum_{i=1}^{s}\frac{1}{x_i} - \sum_{i=1}^{s}\ln(1-e^{-\beta/x_i}) + \alpha\Im(\beta).$$

where, $s=m,\ \Im(\beta)=\sum_{i=1}^{m}(R_i+1)\ln(1-e^{-\beta/x_i})$ and $s=J,\ \Im(\beta)=\sum_{i=1}^{s}(R_i+1)\ln(1-e^{-\beta/x_i})+R_J^*\ln(1-e^{-\beta/T})$ for Case I and Case II respectively. Taking derivatives with respect to α and β from log-likelihood function and equating them to zero, we obtain the maximum-likelihood estimate of the parameter α as $\hat{\alpha}_{ML}=\frac{-s}{\Im(\beta)}$. The MLE of β can be obtained by solving the non-linear equation

$$h(\beta) = \frac{s}{\beta} - \sum_{i=1}^{s}\frac{1}{x_i(1-e^{-\beta/x_i})} - \frac{s\hbar(\beta)}{\Im(\beta)} = 0 \qquad (2)$$

The solution of Equation (2) can be evaluated numerically by some suitable iterative procedure such as the fixed point iteration (see Panahi and Sayyareh; 2014, 2016).

3 Bayesian Estimation

In this section, we compute the Bayes estimates of the unknown parameters of the $GIE(\alpha,\beta)$ distribution under the squared error loss (SEL) function. The independent prior distributions for α and β are taken to be $Gamma(a,b)$ and $Gamma(c,d)$ respectively. Unfortunately, we cannot compute the Bayes estimate of any function of α and β say $g(\alpha,\beta)$, under squared error loss function analytically. Therefore, we adopted the useful approximation namely Lindley's approximation to compute them.

Lindley's Approximation: Lindley (1980) first proposed his procedure to approximate any ratio of integrals. Based on Lindley's approximation, the approximate Bayes estimates of α and β for the squared error loss function become

$$\hat{\alpha}_{SEL} = E(\hat{\alpha}\,|data) = \hat{\alpha} + \frac{1}{2}\Big[2(\frac{a-1}{\hat{\alpha}}-b)(\frac{\Im_2}{\Im_1\Im_2-(\Im_3)^2}) - 2(\frac{c-1}{\hat{\beta}}-d)(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2})$$

$$+\left(\frac{\Im_2}{\Im_1\Im_2-(\Im_3)^2}\right)^2\hat{L}_{\alpha\alpha\alpha}+\left(\frac{\Im_2}{\Im_1\Im_2-(\Im_3)^2}\right)\left(\frac{\Im_1}{\Im_1\Im_2-(\Im_3)^2}\right)\hat{L}_{\beta\beta\alpha}$$

$$+2\left(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2}\right)\left(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2}\right)\hat{L}_{\alpha\beta\beta}-\left(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2}\right)\left(\frac{\Im_1}{\Im_1\Im_2-(\Im_3)^2}\right)\hat{L}_{\beta\beta\beta}\Bigg]$$

and

$$\hat{\beta}_{SEL}=E\left(\hat{\beta}\,|data\right)=\hat{\beta}+\frac{1}{2}\Bigg[2\left(\frac{c-1}{\hat{\beta}}-d\right)\left(\frac{\Im_1}{\Im_1\Im_2-(\Im_3)^2}\right)-2\left(\frac{a-1}{\hat{\alpha}}-b\right)\left(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2}\right)$$

$$+\left(\frac{\Im_1}{\Im_1\Im_2-(\Im_3)^2}\right)^2\hat{L}_{\beta\beta\beta}-3\left(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2}\right)\left(\frac{\Im_1}{\Im_1\Im_2-(\Im_3)^2}\right)\hat{L}_{\alpha\beta\beta}-\left(\frac{\Im_2}{\Im_1\Im_2-(\Im_3)^2}\right)\left(\frac{\Im_3}{\Im_1\Im_2-(\Im_3)^2}\right)\hat{L}_{\alpha\alpha\alpha}\Bigg]$$

respectively. Where, $\hat{\alpha}$ and $\hat{\beta}$ are the MLE of α and β, respectively and $\Im_1=\frac{-s}{\hat{\alpha}^2}$; $\Im_2=-\frac{s}{\hat{\beta}^2}-\sum_{i=1}^{s}\frac{e^{-\hat{\beta}/x_i}}{x_i^2(1-e^{-\hat{\beta}/x_i})^2}-\hat{\alpha}\Re(\hat{\beta})$; $\Im_3=\hbar(\hat{\beta})$; $\hat{L}_{\alpha\alpha\alpha}=\frac{\partial^3 L(\alpha,\beta)}{\partial\alpha^3}\Big|_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$; ,$\hat{L}_{\beta\beta\beta}=\frac{\partial^2 L(\alpha,\beta)}{\partial\beta^2}\Big|_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$, ..., where, $\Re(\hat{\beta})=\sum_{i=1}^{s}\frac{e^{-\hat{\beta}/x_i}(1+R_i)}{x_i^2(1-e^{-\hat{\beta}/x_i})^2}$;

$s=m$ and $\Re(\hat{\beta})=\sum_{i=1}^{s}\frac{e^{-\hat{\beta}/x_i}(1+R_i)}{x_i^2(1-e^{-\hat{\beta}/x_i})^2}+\frac{R_J^*e^{-\hat{\beta}/T}}{T^2(1-e^{-\hat{\beta}/T})^2}$; $s=J$ for Case I and Case II respectively. Also, $\hbar(\beta)$ is defined before.

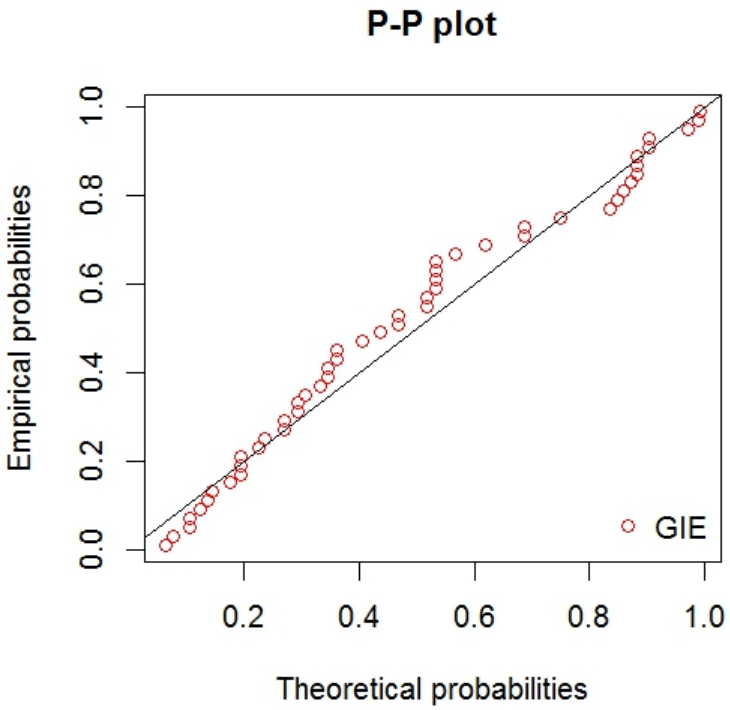


Figure 1: The P-P plot for the data.

4 Real Data Analysis

Previous sections dealt with the analytical technique and this section focuses on the numerical one through practical data set, which was originally considered by Lieblein and Zelen (1956).

The following data set represents the number of millions of revolutions before failure for each of the 23 ball bearings in a life test 17.88, 28.92, 33.0, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4. To check for goodness of fit, we provide the P-P plot in Figure 1. Also we compute the Kolmogorov-Smirnov distance, it is 0.0918 and the associated p-value is 0.9902. Therefore, based on the p-value we can say that the GIE distribution fits quite well to the above data more than any other distribution. Now, we created an artificial data by progressive Type II hybrid censoring. In this case we have $n = 23$ and we took$m = 10$, $T = 85$, $R_1 = ... = R_9 = 1$, $R_{10} = 4$. From the sample corresponding to Case I (17.88, 28.92, 41.52, 45.60, 48.40, 51.96, 54.12, 67.80, 68.64, 84.12), we obtain the MLEs and the BELs of α and β as (10.7152, 129.727) and (9.654, 126.323), respectively. Now consider $m = 10$ and $T = 60$ ($J = 7$) and $R_i's$ to be the same as before. It is observed that the MLEs and the BELs of α and β as (21.722, 136.021) and (20.501, 134.084) respectively.

References

E. Bayat Mokhtari, A. Habibi Rad, F. Yousefzadeh, Inference for Weibull distribution based on progressively Type-II hybrid censored data, J. Stat. Plan. Infer., 141 (2011), 2824-2838.

D. Kundu, and A. Joarder, Analysis of Type-Ii progressively hybrid censored data, Comput. Stat. Data Analysis, 50(2006), 2509- 2528.

D.V. Lindley, Approximate Bayesian method, Trabajos de Estadistica, 31(1980), 223-245.

H. Panahi, A. Sayyareh, Parameter estimation and prediction of order statistics for the Burr Type XII distribution with Type II censoring, J. Appl. Stat., 41(2014), 215-232.

H. Panahi, A. Sayyareh, Estimation and prediction for a unified hybrid-censored Burr Type XII distribution, J. Stat. Comput. Simul., 86(2016), 55-73.

J. Lieblein and M. Zelen, Statistical investigation of the fatigue life deep groove ball bearings, Journal of Research of the National Bureau of Standards, 57 (1956), 273316.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Optimal mixed redundancy allocation in a bridge network by a hierarchical memetic algorithm

Ramezani, Z. [1] and Pourdarvish, A. [2]

[1,2] Department of Mathematics and Statistics, Mazandaran University, Babolsar, Iran

Abstract

Optimization of redundancy problems in reliability by meta-heuristic methods has attracted attention of many researchers during the last few years. In this paper, we have used hierarchical memetic algorithm (HMA) to optimize the mixed redundancy of a bridge network under cost constraint in minimal paths to increase the efficiency of the reliability. We have applied mixed cold and hot redundancies for all levels (components, subsystems and system) simultaneously in the network. What is interesting in this paper is that there is no limitation on the type of distribution of each component. Therefore, we try to optimize mixed redundancy for all the levels and find the maximum reliability for different costs in a network. Finally, the numerical results have been derived by HMA.

Keywords: Hierarchical memetic algorithm, Mixed redundancy allocation, A bridge network, minimal paths

1 Introduction

During the last two decades various optimization methods have been used to tackle redundancy allocation problems. Yun and Kim [11] optimized a restricted redundancy allocation problem (RAP) using genetic algorithm (GA). Yun et al. presented a multi-level redundancy allocation problem (MLRAP) optimized by GA and they succeeded to obtain a constant vector as the optimal solution. Kumar et al. [4] presented a MLRAP in a hierarchical structure, so that redundancy units were allocated to each level and optimized by hierarchical genetic algorithm

[1] zsramezani@gmail.com
[2] a.pourdarvish@umz.ac.ir

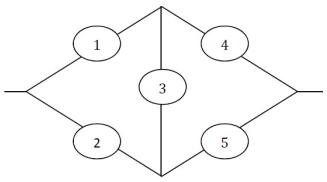


Figure 1: A bridge network

(HGA). Many reliability researches based on meta-heuristic methods have been developed (simulated annealing (Kim et al. [11]), heuristic method ((Safari and Tavakoli [6]),You and Chen [10]),ant colony optimization (Liang and smith [12])). In this paper we have used a multi-level mixed redundancy allocation problem (MLMRAP) to maximize the system reliability, which gives more efficient reliability and better results and a hierarchical memetic algorithm (HMA) is proposed to optimize mixed redundancy in a network. Then it compared to other optimization methods. In this method there is no restriction on selection of distribution of each unit. Therefore, in this article the exponential distribution with different parameters are assumed and the simulated data are tested on a network and using the minimal paths the optimal mixed redundancy is derived. The structure of the paper is as follows: In section 2, we introduce the structure of MLMRAP of a network. In section 3, the optimization of MLMRAP is formulated and the HMA is considered to solve MLMRAP. In section 4, the numerical results on a bridge network are given. In section 5, conclusion is presented.

2 Problem description

In this paper, we apply a new method using minimal paths to obtain the optimal mixed hot and cold redundancy allocation in different levels of component, subsystem and system in a network. A path collection "p" is a set of components u which mentions the operation of the whole system i. e. the system would work if all path units work. The path set is presumed to be minimal, if no other path is formed. In a bridge network in Figure2 minimal paths are as shown below:

$$p_1 = \{u_1, u_4\},\ p_2 = \{u_2, u_5\},\ p_3 = \{u_1, u_3, u_5\},$$
$$p_4 = \{u_2, u_3, u_4\}$$

We know a bridge network system does work if and only if at least one of the series minimal paths is working. Therefore, this structure may be constituted of a parallel system of minimal paths connected in series. In this paper, algorithm is designed for each component distribution. So, reliability function is defined for each unit when there are $i+1$ hot and $n-i-1$ cold redundancy numbers as follows [1]:

$$\begin{aligned} R(t) &= 1 - \int_0^t \int_0^t \cdots \int_{\tau_i}^t \cdots \int_{\tau_{n-2}}^t \int_{\tau_{n-1}}^t f_1(\tau_1) f_2(\tau_2) \cdots \\ &\quad f_{i+1}(\tau_{i+1}) \cdots f_{n-1}(\tau_{n-1} - \tau_{n-2}) f_n(\tau_n - \tau_{n-1}) \\ &\quad d\tau_n d\tau_{n-1} \cdots d\tau_2 d\tau_1 \end{aligned}$$

For example, the reliability function of a system component with two hot and two cold redundancies are calculated by:

$$\begin{aligned} R(t) &= 1 - \int_0^t \int_{\tau_1}^t \int_{\tau_2}^t \int_{\tau_3}^t f_1(\tau_1) f_2(\tau_2) f_3(\tau_3 - \tau_2) \\ &\quad f_4(\tau_4 - \tau_3) d\tau_4 d\tau_3 d\tau_2 d\tau_1 \end{aligned}$$

We are going to optimize MLMRAP in the network. Therefore, obtaining maximum improved reliability and optimal redundancy number based on cost constraint are calculated by:

$$\begin{array}{cc} \max & R_{sys}(t) = R(h,c) \\ \text{subject to} & C_{sys} = C(h,c) \leq C_0 \end{array}$$

Where C_0 is constant cost. We can obtain cost of each unit in the multi-level network (For above example) as follows:

$$C_i = \begin{cases} \sum_{j=1}^{n_i} \sum_{s=1}^{z_{ij}} C_{ij}^s, & \text{if } x_i \text{ is not at the second lowest level} \\ \sum_{j=1}^{n_i} \sum_{s=1}^{z_{ij}} C_{ij}^s + a_{ij}^{z_{ij}}, & \text{if } x_i \text{ is at the second lowest level} \end{cases}$$

Where $z_{ij} = h_{ij} + c_{ij}$ is redundancy number of ijth unit at the lowest level. Also, h, c are the symbols for a hot and cold redundancy vector, respectively, and additional costs of components at the second lowest level are presented with the symbol a_{ij}. The mixed hot and cold redundancies are allocated in the component, subsystem and system units simultaneously. For example, x_i, x_{ij} and x_{ijk} units are system, subsystem and component units, respectively as 1, 4 components in 1 subsystem are shown with x_{111}, x_{114} and 2, 5 components in 2 subsystem are shown by x_{122}, x_{125} and 1, 3 and 5 components in 3 subsystem are shown by x_{131}, x_{133}, x_{135} and 2, 3 and 4 components in 4 subsystem are shown by x_{142}, x_{143} and x_{144}. Also, h_i, c_i, h_{ij}, c_{ij}, h_{ijk} and c_{ijk} are hot and cold redundancy numbers of system, subsystem and component, respectively and z is summation of hot and cold redundancy numbers of each unit in each level(the total of redundancies for every unit).

3 Hierarchical memetic algorithm

HMA is used to optimize MLMRAP includes the following steps:

3.1 Encoding solution

The initial population is randomly generated and included the coded information of decision parameters. The encoding process starts from system level the first and iterates to reach the lowest second level, so that the hot and cold redundancy numbers for each unit of the level is specified. The redundancy numbers of hot and cold change respectively from 1 to p and 0 to p, where p is pre-determined constant.

3.2 Evaluation

We compute the reliability of each individual and take it as an objective function and use the penalty function suggested by Gen and Cheng [3] consider if it falls out of the cost constraint or not. To evaluate the quality of the solution we use the equation 3.1. Where $R(h,c)$, $\psi(h,c)$, h and c are respectively reliability function, penalty function and vectors of hot and cold decision variables.

$$f(h,c) = R(h,c) * \psi(h,c) \tag{3.1}$$

3.3 Genetic operator

The two genetic operators which have been applied in this theory are called a first breadth crossover and a first breadth mutation. The crossover operator for a hierarchical system between

Table 1: Input data

unit	λ	cost	a	unit	λ	cost	a	unit	λ	cost	a
1	12.018	1	1	3	13.02	1	2	5	11.125	2	2
2	10.01	2	2	4	11.02	2	2				

two individual executes in two steps. In the first step we have chosen a subsystem at random, where all the units at the same level have the chance to be chosen. In the second step the selected units are exchanged with their inferior structure. Finally, these operations render to producing of two new individuals. This is called the first breadth crossover. This process is repeated for the mutation operator also, with the difference that in the second step the hot and cold redundancy number of the selected units are exchanged with the chosen integers randomly. Changing the redundancy units number, changes the lower unit structure. This is called the first breadth mutation.

3.4 Local search

In this paper, A heuristic local search is applied which consists of three steps. In the first step all the individuals are evaluated by:

$$\gamma(h,c) = \frac{R(h,c)}{C(h,c)} \tag{3.2}$$

Then the individual with largest $\gamma(h,c)$ is selected for local search. This strategy is used in MLMRAP, so that to have the highest reliability taking in view the cost constraint. Therefore, this criterion is used to measure an individual, as the minimum cost is considered. The second step of the local search operator is performed on the selected individual. A subsystem of this individual randomly is chosen and a unit of the chosen level is selected. The local search changes the system structure partially, and then it is used for the component level only. The third step is to select 10 pairs of the components randomly and changes the hot and cold redundancy numbers of each component at random. The redundancy number of a unit reduces from a pair to one and obtains two new individuals. Then the redundancy of a component is reduced from a pair to one unit and the redundancy of the other component is increased by one and vice versa. Two new individuals are obtained based on the same procedure. Again, we consider the selected individual and this time the hot or cold redundancy number of each pair components is increased by one and shall reach to two new individuals. In every loop of the local search algorithm 4 new individuals are produced for each pair of components. These 4 new individuals have been checked. So that to satisfy the cost constraint. The new individuals are added to a provisional archive of the feasible solutions and the local search algorithm is repeated for every 10 pairs.

3.5 Evolution

In this part, all the initial and new individuals are checked and they are put in an ascending order and a new population is selected equal to initial population of the best individuals and HMA process is repeated for the new population until the time to reach to the predetermined generation number as the stop alarm, and the last individual with the maximum mixed redundancy reliability in the multi-level system is achieved.

4 Experimental results and discussion

The input data for the system reliability optimization problem are summarized in Table 1. The input data consists of failure rate (λ), corresponding cost and additional cost for each unit.

Table 2: Optimal mixed redundancy allocation in a bridge network system using HMA

	Cost constraint										
redundancy	550	560	570	580	590	600	610	620	630	640	650
(h_1, c_1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
(h_{11}, c_{11})	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(1, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
(h_{12}, c_{12})	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
(h_{13}, c_{13})	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
(h_{14}, c_{14})	(1, 0)	(1, 0)	(1, 0)	(2, 0)	(2, 0)	(2, 0)	(1, 0)	(1, 0)	(1, 0)	(2, 0)	(2, 0)
h_{111}	2 1	2 1	2 1	2 2	2 2	2	2 1	2 1	2 1	2 2	2 2
c_{111}	2 1	2 1	2 1	2 1	2 1	2	3 1	3 1	3 1	2 1	2 1
h_{114}	2 1	2 1	2 1	2 2	2 2	2	3 1	3 1	3 1	2 2	2 2
c_{114}	0 0	0 0	0 0	2 0	2 0	2	2 0	2 0	2 0	2 0	2 0
h_{122}	2 1	2 1	2 1	2 2	2 2	2 2	1 1	1 1	1 1	2 1	2 1
c_{122}	1 1	1 1	1 1	1 2	1 2	2 1	2 1	2 1	2 1	1 1	1 1
h_{125}	2 1	2 1	2 1	2 2	2 2	2 2	3 1	3 1	3 1	2 1	2 1
c_{125}	1 1	1 1	1 1	1 1	1 1	2 2	1 1	1 1	1 1	1 1	1 1
h_{131}	1 1	1 1	1 1	2 1	2 1	3 1	2 1	2 1	2 1	2 1	2 1
c_{131}	2 2	2 2	2 2	1 1	1 1	1 1	1 2	1 2	1 2	1 1	1 1
h_{133}	2 1	2 1	2 1	2 1	2 1	2 1	2 1	2 1	2 1	2 1	2 1
c_{133}	2 1	2 1	2 1	2 1	2 1	0 0	2 1	2 1	2 1	2 1	2 1
h_{135}	1 1	1 1	1 1	2 2	2 2	2 1	2 1	2 1	2 1	2 1	2 1
c_{135}	1 2	1 2	1 2	2 2	2 2	0 0	1 1	1 1	1 1	1 1	1 1
h_{142}	2	2	2	2 2	2 2	2 1	2	2	2	2 2	2 2
c_{142}	0	0	0	0 0	0 0	0 1	1	1	1	1 1	1 1
h_{143}	3	3	3	2 1	2 1	2 1	1	1	1	2 1	2 1
c_{143}	1	1	1	2 0	2 0	0 1	2	2	2	1 2	1 2
h_{144}	2	2	2	2 2	2 2	2 1	1	1	1	2 2	2 2
c_{144}	0	0	0	0 1	0 1	1 1	1	1	1	1 2	1 2
R_{sys}	0.9978	0.9978	0.9978	0.9987	0.9987	0.9991	0.9994	0.9994	0.9994	0.9996	0.9996
cost	159	159	159	260	260	202	214	214	214	282	282

Table 3: Optimal mixed redundancy allocation in a bridge network system using HGA

	Cost constraint										
redundancy	550	560	570	580	590	600	610	620	630	640	650
(h_1, c_1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
(h_{11}, c_{11})	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
(h_{12}, c_{12})	(2, 0)	(1, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
(h_{13}, c_{13})	(2, 0)	(1, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
(h_{14}, c_{14})	(2, 0)	(2, 0)	(2, 0)	(1, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)	(2, 0)
h_{111}	2 2	2 1	2 1	2 1	1 1	1 1	2 2	2 1	2 2	2 2	1 1
c_{111}	1 1	2 2	0 2	2 1	1 0	1 0	1 1	0 1	2 1	2 1	2 1
h_{114}	1 2	2 1	3 1	2 2	2 2	2 2	2 2	2 1	2 2	2 2	1 2
c_{114}	2 1	1 1	0 1	0 0	1 0	1 0	2 1	0 0	2 0	2 0	2 1
h_{122}	2 2	2	1 1	2 1	2 2	2 2	2 1	2 1	2 2	2 2	2 1
c_{122}	2 2	1	2 2	2 2	2 1	2 1	1 2	2 1	1 2	1 2	2 1
h_{125}	2 1	2	2 2	2 1	2 1	2 1	1 1	2 1	2 2	2 2	2 1
c_{125}	1 1	1	1 1	1 1	0 1	0 1	2 2	2 2	1 1	1 1	2 0
h_{131}	1 1	2	2 1	1 1	2 1	2 1	2 2	2 2	2 1	2 1	2 1
c_{131}	1 1	1	1 1	2 2	2 2	2 2	1 2	2 2	1 1	1 1	2 1
h_{133}	1 1	2	1 1	2 1	2 1	2 1	2 1	1 1	2 1	2 1	2 1
c_{133}	1 1	1	1 1	1 1	2 1	2 1	1 1	1 1	2 1	2 1	2 1
h_{135}	2 1	2	2 2	1 2	2 1	2 1	2 2	1 2	2 2	2 2	2 2
c_{135}	2 1	1	1 1	2 1	1 2	1 2	2 1	1 2	2 2	2 2	2 1
h_{142}	1 1	2 1	2 2	2	2 1	2 1	2 1	1 1	2 2	2 2	2 2
c_{142}	2 1	2 0	2 1	2	2 2	2 2	1 2	2 1	0 0	0 0	2 2
h_{143}	1 1	2 2	2 1	2	2 2	2 2	2 1	1 1	2 1	2 1	2 2
c_{143}	2 1	0 1	1 1	1	1 2	1 2	2 1	2 1	2 0	2 0	2 1
h_{144}	2 2	2 2	1 1	2	2 2	2 2	2 1	1 1	2 2	2 2	2 1
c_{144}	2 1	1 0	2 2	1	1 2	1 2	1 1	2 1	0 1	0 1	2 1
R_{sys}	0.9766	0.9953	0.9955	0.9959	0.9978	0.9978	0.9983	0.9985	0.9987	0.9987	0.9994
cost	249	214	243	220	251	251	264	221	260	260	284

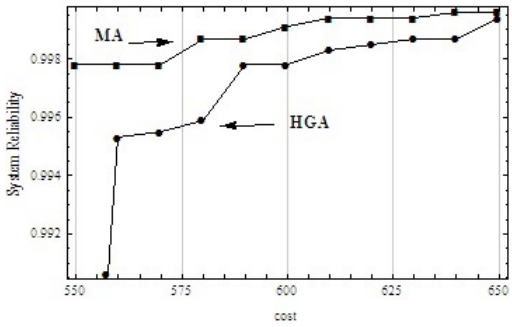


Figure 2: The optimal reliability function by the MA and HGA under different constraint values

The objective is to maximize system reliability with a mixture of hot and cold redundancies for all levels simultaneously in the network, under system cost constraint. We apply HGA and HMA presented in the previous section to solve our system reliability, and then the results are calculated and compared. This numerical example consists of 2 redundancy maximum for the subsystem units and 2 redundancy maximum for the component units. The system has five components. The system cost constraint has ranges from 550 to 650. For each constraint, every algorithm runs 10 times independently. A single case study, the performance over different cost constraint values, and the performance on test instance with different system parameters. The maximum generation number was set to 500 for both methods. The crossover rate p_c and the mutation rate p_m are set to 0.8 and 0.05. An initial population of 100 individuals is generated, 50 and 10 offsprings are generated by crossover and mutation operators, respectively. The simulation process can also be applied for other maximum redundancies. For solving MLMRAP by HMA, the solution is clearly improved when adding the local search algorithm but HGA does not consist of the local search algorithm and other steps of algorithm performance are the same in both algorithms. In the case of network, the influence of the local search is clearly highlighted. The optimal solutions have a mixture of hot and cold redundancy indicating the fact that the reliability of the system design with a mixture of hot and cold standby redundancy is more than those of systems designed exclusively with either hot or cold standby redundancy with the same system cost constraint.

In Tables 2 and 3, the best results obtained by the HMA are slightly better than those obtained by HGA. The proposed HMA is an effective algorithm for solving the reliability redundancy optimization problems, and it performs better than or competitive to HGA existing algorithm in the literature. Figure 4 confirms these results.

5 Conclusion

A design was presented to determine optimal solutions for the mixed hot and cold redundancy allocation problem for a bridge network system. We applied mixed cold and hot redundancy for all the levels (components, subsystems and system) simultaneously in the network by the HMA. The efficiency of the reliability increased when we used the minimal paths for redundancy optimization. Also, superiority of the HMA was specified by comparing of the HMA and the HGA.

References

[1] Boddu, P. and Xing, L. (2012), Redundancy allocation for k-out-of-n: G systems with mixed spare types, *Proceedings of the 58th annual reliability and maintainability symposium*, Reno, NV, USA, January.

[2] Coit, D.W. and Smith, A.E. (1996), Reliability optimization of series-parallel systems using a genetic algorithm, *IEEE Transactions on Reliability*, **45**, 254-260.

[3] Gen, M. and Cheng, R. (1996), Survey of penalty technique in genetic algorithms, *International Conference on Evolutionary Computation*, Japan, 804-809.

[4] Kumar, R., Izui, K., Yushimura, M. and Nishiwaki, S. (2009), Multi-level redundancy allocation optimization using hierarchical genetic algorithm, *IEEE Transactions on Reliability*, **57**, 650-661.

[5] Liang, Y. C. and Smith, A. (2004), An ant colony optimization algorithm for the redundancy allocation problem (RAP), *IEEE Transactions on Reliability*, **53**, 417-423.

[6] Safaei, N., Tavakkoli-Moghaddam, R. and Kiassata, C. (2012), Annealing-based particle swarm optimization to solve the redundant reliability problem with multiple component choices, *Applied Soft Computing*, **12** 3462-3471.

[7] Tavakkoli-Moghaddam, R., Safari, J. and Sassani, F. (2007), Reliability optimization of series-parallel systems with a choice of redundancy strategies using a genetic algorithm, *Reliability Engineering and System Safety*, **93** (4), 550-556.

[8] Wang, Z., Tang, K. and Yao, X. (2010), A memetic algorithm for multi-level redundancy allocation, *IEEE Transactions on Reliability*, **59** (4), 754-765.

[9] Yoshimura, M. and Izui, K. (2002), Smart optimization of machine systems using hierarchical genotype representations, *ASME Journal of Mechanical Design*, **124**(3), 375-384.

[10] You, P.S. and Chen, T. C. (2005), An efficient heuristic for series-parallel redundant reliability problems, *Computers and Operations Research*, **32**(8), 375-384.

[11] Yun, W. Y. and Kim, J. W. (2004), Multi-level redundancy optimization in series systems, *Computers and Industrial Engineering*, **46**, 337-346.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

On the reliability of series and parallel systems with randomized components

Rahimi, Z. [1] and Zarezadeh, S. [2] and Sharafi, M. [3]

[1,2,3] Department of Statistics, Shiraz University, Shiraz 71454, Iran

Abstract

In this paper, we consider a series or parallel system of independent components and assume that the components are selected from three distinct batches. Under some conditions, it is proved that if the number of components chosen from the first and the second batches increases in the usual stochastic order then the reliability of the series (parallel) system increases (decreases) in the same stochastic order sense.

Keywords: Reliability, stochastic ordering, series system.

1 Introduction

One of the significant problem in reliability is optimization of complex system; see, e.g., [3] and references therein. Barlow and Proschan [1] gave some examples of bounds for the reliability of series and parallel systems when the number of components is a random variable. Li and Hu [4] studied stochastic comparisons between the lifetimes of a series system with redundant components and two allocation choices.

Di Crescenzo and Pellerey [2] considered a series or parallel system whose components are chosen from two distinct batches. They showed that the reliability increases if the the number of components chosen from the first batch increases in convex order. In this paper, we explore this result when the components of the system may be chosen from more than two batches. For example, assume that there are three different batches B_X, B_Y, B_Z and the corresponding lifetimes in each batch are $\{X_i; i = 1, ..., n\}$, $\{Y_i; i = 1, ..., n\}$ and $\{Z_i; i = 1, ..., n\}$.

[1]zahrarahimi268@gmail.com
[2]s.zarezadeh@shirazu.ac.ir
[3]msharafi@susc.ac.ir

Let $X_1, X_2, \ldots, X_n$, $Y_1, Y_2, \ldots, Y_n$ and $Z_1, Z_2, \ldots, Z_n$ be independent and identically distributed as random variables X, Y and Z, respectively. Suppose the components of a series or parallel system with n components are chosen from batches B_X, B_Y, B_Z. If k components is selected from batch B_X and l components from batch B_Y and $n-k-l$ components are taken from the batch B_Z, $0 \leq k+l \leq n$, define

$$\Pi_{(k,l)} = \begin{cases} Z_1, ..., Z_n & \text{if } k=0, l=0 \\ Y_1, ..., Y_n & \text{if } k=0, l=n \\ X_1, ..., X_n & \text{if } k=n, l=0 \\ & \text{if } k=1, ..., n-1, \\ X_1, ..., X_k, Y_{k+1}, ..., Y_{l+k}, Z_{l+k+1}, ..., Z_n & \qquad\qquad l=1, ..., n-1. \end{cases}$$

By this setting, the lifetimes of the series and parallel systems can be respectively represented as $T^s_{(k,l)} = \min\{\Pi_{(k,l)}\}$ and $T^p_{(k,l)} = \max\{\Pi_{(k,l)}\}$. The random variables K and L denote the number of components selected from the first and the second batches, respectively.

The aim of this paper is to survey the appropriate conditions under which the reliability of the system is improved. Under some conditions, it is proved that if (K, L) increases in the usual stochastic order then the reliability of the series system increases and under same assumptions the reliability of the parallel system decreases. This result can be generalized for the case where we have m different batches. Throughout the paper, for any random variables W we denote its distribution function by $F_W(w) = P(W \leq w)$ and its reliability function by $\bar{F}_W(w) = P(W > w)$.

2 Series system

Consider a series system with n independent components. Suppose the components of the system are chosen from three different batches in which the first batch contains the components with common reliability function $\bar{F}_X$, the second includes components with common reliability function $\bar{F}_Y$ and the third has components having identical reliability function $\bar{F}_Z$. Let K and L be discrete random variable which denote the components chosen from the first and the second batch, respectively. If $T^s_{(K,L)}$ denotes the lifetime of the series system, then it is easy to show that the reliability of the system can be written as

$$P(T^s_{(K,L)} > t) = \sum\sum_{(k,l)\in S_{(K,L)}} \bar{F}^k_X(t)\bar{F}^l_Y(t)\bar{F}^{n-k-l}_Z(t)P(K=k, L=l).$$

where $S_{(K,L)} = \{(k,l)|k = 0, 1, ..., n, l = 0, 1, ..., n, 0 \leq k+l \leq n\}$ is the support of joint random variable (K, L).

The following theorem is a main tool to obtain the next results (Theorem 6.B.18 of [5]).

Theorem 1. *Let $\{G^i_\theta, \theta \in \mathcal{X}_i\}$, $i = 1, 2, ..., n$, be n families of univariate distribution functions as above. Let Θ_1 and Θ_2 be two random vectors with supports in $\Pi^n_{i=1}\mathcal{X}_i$ and distribution functions F_1 and F_2, respectively. Let Y_1 and Y_2 be two random vectors with distribution functions H_1 and H_2 given by*

$$H_j(y_1, y_2, ..., y_n) = \int_{\chi_1}\int_{\chi_2}\ldots\int_{\chi_n} \Pi^n_{i=1}G^i_{\theta_i}(y_i)dF_j(\theta_1, \theta_2, ..., \theta_n)$$

for $(y_1, y_2, ..., y_n) \in \mathbb{R}^n$, $j = 1, 2$. If $X_i(\theta) \leq_{st} X_i(\theta')$ whenever $\theta \leq \theta'$, $i = 1, 2, ..., n$, and $\Theta_1 \leq_{st} \Theta_2$, then $Y_1 \leq_{st} Y_2$.

Theorem 2. *Let $Z \leq_{st} X$ and $Z \leq_{st} Y$. If $(K_1, L_1) \leq_{st} (K_2, L_2)$, then $T^s_{(K_1,L_1)} \leq_{st} T^s_{(K_2,L_2)}$.*

Proof. Define a two-variate function as follows

$$Q(k,l) = \left(\frac{\bar{F}_X(t)}{\bar{F}_Z(t)}\right)^k \left(\frac{\bar{F}_Y(t)}{\bar{F}_Z(t)}\right)^l.$$

By the assumptions of the theorem, it can be concluded that $Q(k,l)$ is an increasing function of (k,l). Then, from $(K_1,L_1) \leq_{st} (K_2,L_2)$ we can write

$$E(Q(K_1,L_1)) \leq E(Q(K_2,L_2))$$

Then

$$\sum\sum_{(k_1,l_1)\in S_{(K_1,L_1)}} \left(\frac{\bar{F}_X(t)}{\bar{F}_Z(t)}\right)^{k_1} \left(\frac{\bar{F}_Y(t)}{\bar{F}_Z(t)}\right)^{l_1} P(K_1=k_1,L_1=l_1)$$
$$\leq \sum\sum_{(k_2,l_2)\in S_{(K_2,L_2)}} \left(\frac{\bar{F}_X(t)}{\bar{F}_Z(t)}\right)^{k_2} \left(\frac{\bar{F}_Y(t)}{\bar{F}_Z(t)}\right)^{l_2} P(K_2=k_2,L_2=l_2).$$

By manipulation the expression $(\bar{F}_Z(t))^n$ on two sides of the above inequality, it is followed that $P(T^s_{(K_1,L_1)} > t) \leq P(T^s_{(K_2,L_2)} > t)$ and hence the proof is completed. □

3 Parallel system

Assume a parallel system has n independent components whose components choose from three different batches, B_X, B_Y, B_Z. The components in batches B_X, B_Y, B_Z arev independent and identically distributed with distribution functions F_X, F_Y, F_Z, respectively. Let random variables K and L indicate the number of components chosen from batches B_X and B_Y, respectively. If $T^p_{(K,L)}$ is the lifetime of the parallel system, then

$$P(T^p_{(K,L)} \leq t) = \sum\sum_{(k,l)\in S_{(K,L)}} F^k_X(t)F^l_Y(t)F^{n-k-l}_Z(t)P(K=k,L=l),$$

where $S_{(K,L)} = \{(k,l)|k=0,1,...,n,l=0,1,...,n,0\leq k+l\leq n\}$ is the support of joint random variable (K,L).

Theorem 3. *Let $Z \leq_{st} X$ and $Z \leq_{st} Y$.If $(K_1,L_1) \leq_{st} (K_2,L_2)$, then $T^p_{(K_1,L_1)} \geq_{st} T^p_{(K_2,L_2)}$.*

Proof. Let $G(k,l) = (F_X(t)/F_Z(t))^k (F_Y(t)/F_Z(t))^l$. By the supposition of theorem we can follow that $G(k,l)$ is an decreasing function of (k,l). Then, from $(K_1,L_1) \leq_{st} (K_2,L_2)$ we can write

$$E(G(K_1,L_1)) \geq E(G(K_2,L_2))$$

Then

$$\sum\sum_{(k_1,l_1)\in S_{(K_1,L_1)}} \left(\frac{F_X(t)}{F_Z(t)}\right)^{k_1} \left(\frac{F_Y(t)}{F_Z(t)}\right)^{l_1} P(K_1=k_1,L_1=l_1)$$
$$\geq \sum\sum_{(k_2,l_2)\in S_{(K_2,L_2)}} \left(\frac{F_X(t)}{F_Z(t)}\right)^{k_2} \left(\frac{F_Y(t)}{F_Z(t)}\right)^{l_2} P(K_2=k_2,L_2=l_2)$$

If two sides of the above inequality are manipulated by $(F_Z(t))^n$, the result is obtained. □

References

[1] Barlow, R. E., Proschan, F. (1981), *Statistical theory of reliability and life testing Probability models*, to begin with. MD: Silver Spring.

[2] Di Crescenzo, A. and Pellerey, F. (2011), Stochastic comparisons of series and parallel systems with randomized independent components. *Operations Research Letters*, **39**, 380-384.

[3] Kuo, W. and Rajendra Prasad, V. (2000), An annotated overview of system-reliability Optimization, *IEEE Trans. Reliab*, 176-187.

[4] Li, X. and Hu, X. (2008), Some new stochastic comparisons for redundancy allocations in series and parallel systems, *Statist. Probab. Lett*, **78**, 3388-3394.

[5] Shaked, M. and Shanthikumar, J. G. (2007), *Stochastic Orders*, New York: Springer.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Stochastic Comparisons of $(n-k+1)$-out-of-n Systems with exchangeable components Based on Residual Lifetimes

Salehi, E. [1]

[1] Department of Basic Science, Birjand University of Technology, Birjand, Iran

Abstract

In this paper, we consider a $(n-k+1)$-out-of-n system in which the component lifetimes are exchangeable random variables. In following, we stochastically compare the residual lifetimes of $(n-k+1)$-out-of-n systems in one sample and two samples.

Keywords: inactivity time, stochastic order, dependence, reliability.

1 Introduction

$(n-k+1)$-out-of-n systems are the papular systems of coherent systems. A $(n-k+1)$-out-of-n system is a system consisting of n components and the system works if and only if at least $(n-k+1)$ components out of n components are operating $(k \leq n)$. Two important special cases of $(n-k+1)$-out-of-n systems are parallel systems and series systems corresponding to $k=n$ and $k=1$, respectively. The mean residual lifetime (MRL) function is one of the most important measures in the reliability theory. During the last few years, many authors have investigated properties of MRL for coherent systems specially k-out-of-n systems as well. Among others, we can refer to [1], [2], [3], [5], [10] and [13].

Much attention of authors has been paid to studying properties of the systems restricted to the case when the components are independent. In fact the components are not independent in practical perspectives. The assumption of dependence among components of system is reasonable. In this regard, one can refer to [4], [6], [7], [8], [11] and [12].

In this note, we consider the a $(n-k+1)$-out-of-n system in which the component lifetimes are exchangeable. We present some new stochastic ordering properties of the residual lifetimes of the systems from one sample, and two samples.

[1] salehi@birjandut.ac.ir

2 Preliminaries

In this section, we present the required tools and concepts for the following section. Let T be the lifetime of a coherent system consisting of n components with the lifetimes $T_1, T_2, \ldots, T_n$. Denote by $T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ the ordered lifetimes of the components. If $T_{r:n}$ denotes the rth smallest component lifetime, then the conditional random variable

$$\{T - t | T_{r:n} > t\}, \tag{2.1}$$

represents the residual lifetime of the system under the condition that at least $n - r + 1$ components of the system are working at time t. The other realistic and interesting situation is the residual lifetime of a $(n - k + 1)$-out-of-n system given the condition that at least r components of the system have failed but the system is working at time t, i.e.

$$\{T - t | T_{r:n} \leq t, T > t\}. \tag{2.2}$$

Among the researchers who have extended this concept to the coherent system, we can refer to [2], [8] and [13].

Before giving the main results, we introduce some concepts of stochastic orders which will be used in the next section. For more details see [9].

Definition 1. *Let X and Y be two nonnegative random variables with survival functions $\bar{F}$ and $\bar{G}$, respectively. X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$, if for all t, $\bar{F}(t) \leq \bar{G}(t)$.*

Definition 2. *A density function $f : \mathbf{R^n} \to \mathbf{R}_+$ is said to be multivariate totally positive of order 2 (MTP2) if for all $\mathbf{x}, \mathbf{y} \in \mathbf{R^n}$,*

$$f(\mathbf{x})f(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y})f(\mathbf{x} \vee \mathbf{y}),$$

where $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$ and $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$, with $x_i \vee y_i = \max\{x_i, y_i\}$ and $x_i \wedge y_i = \min\{x_i, y_i\}$, $i = 1, 2, \ldots, n$.

Theorem 1. *Let $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ be two sequences of (not necessarily independent) random variables such that $(X_1, X_2, \ldots, X_k) \leq_{st} (Y_1, Y_2, \ldots, Y_k)$, $k \geq 1$. Then $X_{i:m} \leq_{st} Y_{j:n}$, whenever $i \leq j$ and $m - i \geq n - j$.*

Let $\{X_1, X_2, \ldots\}$ be a sequence of (not necessarily independent) random variables. Then $X_{i:m} \leq_{st} X_{j:n}$, whenever $i \leq j$ and $m - i \geq n - j$.

3 Main results

In this section, we consider a $(n - k + 1)$-out-of-n system consisting of n components with exchangeable lifetimes $T_1, T_2, \ldots, T_n$. We will stochastically compare the residual lifetimes of the systems that introduced in equations (2.1) and (2.2), in one-sample and two-sample problems (see [8]).

Now, we have the following theorems.

Theorem 2. *For any $t > 0$, and $1 \leq r \leq k < n$,*

$$\{T_{k:n} - t | T_{r:n} > t\} \leq_{st} \{T_{k+1:n} - t | T_{r:n} > t\}.$$

Here, we present that the residual lifetime of the system is stochastically decreasing in r, for any fixed k and n under the certain condition, [8].

Theorem 3. *If the joint density function of the exchangeable random variables $T_1, T_2, \ldots, T_n$ satisfies the MTP2 property, then*

i) for $t > 0$ and $1 \leq r < k \leq n$,

$$\{T_{k:n} - t | T_{r+1:n} > t\} \leq_{st} \{T_{k:n} - t | T_{r:n} > t\};$$

ii) for $t > 0$, $r = 1, 2, \ldots, k-2$ and $k \leq n$,

$$\{T_{k:n} - t | T_{r+1:n} \leq t, T_{k:n} > t\} \leq_{st} \{T_{k:n} - t | T_{r:n} \leq t, T_{k:n} > t\}.$$

Let the joint density function of T_i's $i = 1, 2, \ldots, n$, be MTP_2. Using Theorems 2 and 3,

i) For $1 \leq l \leq r \leq k \leq p \leq n$,

$$E(T_{k:n} - t | T_{r:n} > t) \leq E(T_{p:n} - t | T_{l:n} > t);$$

ii) for $1 \leq l \leq r < k \leq n$,

$$E(T_{k:n} - t | T_{r:n} \leq t, T_{k:n} > t) \leq E(T_{k:n} - t | T_{l:n} \leq t, T_{k:n} > t).$$

Now, consider two $(n-k+1)$-out-of-n systems $\mathcal{S}_1$, and $\mathcal{S}_2$ with exchangeable components $\mathbf{T} = (T_1, T_2, \ldots, T_n)$, and $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$, respectively. The following result shows that, when the component lifetimes of two systems are ordered in terms of usual multivariate stochastic order, then the corresponding systems are stochastically ordered in terms of their residual lifetimes (see [8]).

Theorem 4. *If $\{\mathbf{T} | \mathbf{T} \in E_i(t)\} \leq_{st} \{\mathbf{Z} | \mathbf{Z} \in E_i(t)\}$ for $i = 0, 1, \ldots, r-1$ and for $E_i(t) = [0,t]^i \times (t, \infty)^{n-i}$ and $[0,t]^i \times [0, \infty) \times (t, \infty)^{n-i-1}$, then for $t > 0$,*

i) $\{T_{k:n} - t | T_{r:n} > t\} \leq_{st} \{Z_{k:n} - t | Z_{r:n} > t\}$,

ii) $\{T_{k:n} - t | T_{r:n} \leq t, T_{k:n} > t\} \leq_{st} \{Z_{k:n} - t | Z_{r:n} \leq t, Z_{k:n} > t\}$.

References

[1] Asadi, M. Bayramoglu, I. (2006), The mean residual life function of a k-out-of-n structure at the system level, *IEEE Trans. Reliab.*, **55**, 314-318.

[2] Kochar, S. Xu, M. (2010), On results lifetimes of k-out-of-n systems with nonidentical components, *Probab. Eng. Inform. Sci.*, **24**, 109-127.

[3] Li, X., Zhao, P. (2006), Some aging properties of the residual life of k-out-of-n systems, *IEEE Trans. Reliab.*, **55**, 535-541.

[4] Navarro, J., Ruiz, J.M. Sandoval, C.J. (2007), Properties of coherent systems with dependent components, *Commun. Statist. Theor. Meth.*, **36**, 175-191.

[5] Navarro, J. Balakrishnan, N. Samaniego, F. J. (2008), Mixture representations of residual lifetimes of used systems, *J. Appl. Probab.*, **45**, 1097-1112.

[6] Rezapour, M. Salehi, E.T. Alamatsaz, M.H. (2013), Stochastic comparison of residual and past lifetimes of $(n-k+1)$-out-of-n systems with dependent components, *Commun. Statist. Theor. Meth.*, 42, 1-15.

[7] Sadegh, M.K. (2011), A note on the mean residual life function of a coherent system with exchangeable or nonidentical components. *J. Statist. Plann. Infer.*, **141**, 3267-3275.

[8] Salehi, E. Hashemi-Bosra, S.Sh. (2015), The mean general residual lifetime of $(n-k+1)$-out-of-n systems with exchangeable components. To appear in *Commun. Statist. Theor. Meth.*

[9] Shaked, M. Shanthikumar, J. G. (2007). *Stochastic Orders and Their Applications*, New York: Springer.

[10] Tavangar, M. (2014), Some comparisons of residual Life of coherent systems with exchangeable components, *Naval Res. Logist.*, **61**, 549-556.

[11] Tavangar, M. Asadi, M. (2015), The Inactivity Time of Exchangeable Components of k-out-of-n Structures, *Sankhya* **136**, 1197-1206.

[12] Zhang, Z. (2010), Ordering conditional general coherent systems with exchangeable components, *J. Statist. Plann. Infer.*, **140**, 454-460.

[13] Zhao, P. Li, X. Balakrishna, N. (2008), Conditional ordering of k-out-of-n systems with independent but non-identical componentts. *J. Appl. Probab.*, **45**, 1113-1125.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Representing the Reliability Function in Terms of Mean Residual Lifetime of Coherent Systems

Mahdi Tavangar [1]

[1,2] Department of Statistics, University of Isfahan, Isfahan, Iran

Abstract

In this talk, we consider a coherent (or mixed) system consisting of n identical components with independent lifetimes. We show that when the underlying distribution function F is absolutely continuous, then it can be univocally determined by some particular mean residual lives or mean inactivity times of the system and their derivatives.

Keywords: Burn-in test, Bivariate degradation, optimal test, Stochastic process, Inverse Gaussian process

1 Introduction

In reliability engineering, k-out-of-n structure play an important role. A technical system has an k-out-of-n structure if it works when at least k of the n components work. Suppose that the lifetimes of the components are described by random variables $T_1, T_2, ..., T_n$. Assume that T_i's are independent and have a common absolutely continuous distribution function $F(t)$. It is known that the lifetime of the $(n-k+1)$-out-of-n system is represented by the corresponding kth order statistic $T_{k:n}$ Many properties and applications of this system are obtained in the literature (see, for example, [4]). The class of k-out-of-n systems is a special case of a class of systems which is known in the literature as the coherent systems. A structure consisting of n components is known as a coherent system if it has no irrelevant component and the system is monotone in every component. Samaniego [6] showed that the reliability function of the system lifetime T_S can be written as a mixture of the reliability functions of $T_{i:n}$ with weights $s_1, s_2, ..., s_n$ such that $s_i = P\{T_S = T_{i:n}\}$, $i = 1, 2, ..., n$.

[1] m.tavangar@stat.ui.ac.ir

Boland and Samaniego [5] introduced the notion of mixed systems by admitting signatures with arbitrary non-negative coordinates summing up to 1. In other words, any probability vector $(s_1, s_2, ..., s_n)$ is the signature of a mixed system. The mixed system is a randomly chosen k-out-of-n system in which the choice probability is determined by the signature.

In recent years, the residual lifetime of the system at the system level has aroused the interest of many authors. One can define the MRL function of a $(n-k+1)$-out-of-n under the condition that at most $(r-1)$ components of the system have failed, as follows:

$$H_{r,k,n}(t) = E(T_{k:n} - t | T_{r:n} > t), \quad 1 \leq r \leq k \leq n.$$

Special cases of this function are studied, e.g., in [2] and [3].

In recent literature, the inactivity time of a component or a system is also considered by researchers. Suppose that an n-component system is put in use at time $t = 0$. Assume that the system is not monitored continuously. A problem of interest is to get information about the history of the system, e.g., when the components of the system have failed. Motivated by this, Asadi [1] defined and investigated the concept of mean inactivity time (MIT) of a parallel system, at the system level. Tavangar and Asadi [8] have extended the Asadi's results to $(n-k+1)$-out-of-n structures and defined the MIT of such a system as follows,

$$M_{r,k,n}(t) = E(t - T_{r:n} \mid T_{k:n} \leq t), \quad 1 \leq r \leq k \leq n. \tag{1.1}$$

The aim of the present talk is to summarize some representation results for the parent distribution of components of the system in terms of some particular MRL or MIT functions.

2 Main results

Theorem 1. *(Tavangar [7]) Let the components of the system be independent and have a common absolutely continuous distribution F which is strictly increasing on $(0, \omega(F))$, where $\omega(F) = \sup\{x : F(x) < 1\}$. The reliability function $\bar{F}_{T_{r:n}}(t)$ of $T_{r:n}$ can be represented as follows,*

$$\bar{F}_{T_{r:n}}(t) = \exp\left\{ - \int_0^t \frac{1 + \frac{d}{dx} H_{r,k,n}(x)}{H_{r,k,n}(x) - H_{1,k-r,n-r}(x)} dx \right\}, \quad t \in (0, \omega(F)).$$

It is known that there is a one-to-one relation between the distribution function of $T_{r:n}$ and the parent distribution function $F(t)$. Hence we have shown, in Theorem 1, that when $F(t)$ is absolutely continuous and strictly increasing on its support, then it can be uniquely determined by $H_{r,k,n}(t)$ and $H_{1,k-r,n-r}(t)$, for fixed values of r, k, n, $1 \leq r \leq k \leq n$. It is an open question if $H_{r,k,n}(t)$ for specific values of r, k, n uniquely identifies $F(t)$.

A similar result as that of Theorem 1 has already been proved by Tavangar and Asadi [8] for the MIT of the components of k-out-of-n system at the system level. Their result is as follows,

$$F_{T_{k:n}}(t) = \exp\left\{ - \int_t^{\omega(F)} \frac{1 - \frac{d}{dx} M_{r,k,n}(x)}{M_{r,k,n}(x) - M_{r,k-1,k-1}(x)} dx \right\}, \quad t \in (0, \omega(F)),$$

where $M_{r,k,n}(x)$ is the MIT function defined in (1.1). Similar expressions for the parent distribution in terms of conditional MRL and conditional MIT functions and their derivatives are obtained in [9].

Consider the coherent system $\mathcal{S}$ with the property that, with probability 1, it is operating as long as $(n-k+1)$ components operate. Such system must have a signature of order n of the form

$(0, ..., 0, s_k, s_{k+1}, ..., s_n)$, where $s_k > 0$. It is known that the MRL $H^S_{r,n}(t) = E(T_S - t \mid T_{r:n} > t)$ of the coherent system can be represented as follows,

$$H^S_{r,n}(t) = \sum_{i=k}^{n} s_i H_{r,i,n}(t).$$

In the literature, there exist other definitions of the MRL. For example, one can consider the MRL $\tilde{H}^S_{r,n}(t)$ of the coherent system $\mathcal{S}$ with signature $(0, ..., 0, s_k, s_{k+1}, ..., s_n)$,$s_k > 0$, under the condition that exactly $(n - r)$, $r = 1, 2, ..., k - 1$, components of the system are working, and the other components have already failed; that is

$$\tilde{H}^S_{r,n}(t) = E(T_S - t \mid T_{r:n} \le t < T_{r+1:n}).$$

Similarly, we can define the MIT $\tilde{M}^S_{k,n}(t)$ of the coherent system $\mathcal{S}$ with signature $(s_1, s_2, ..., s_r, 0, ..., 0)$, $s_r > 0$, under the condition that exactly $(k - 1)$, $k = r + 1, ..., n$, components have already failed; that is

$$\tilde{M}^S_{k,n}(t) = E(t - T_S \mid T_{k-1:n} \le t < T_{k:n}).$$

The following theorem states that the parent distribution can be uniquely determined from $H^S_{r,n}(t)$ and $\tilde{H}^S_{r,n}(t)$, for some fixed $1 \le r \le k$.

Theorem 2. *(Tavangar, [7]) Let the components of the coherent system $\mathcal{S}$ be independent and have a common absolutely continuous distribution F which is strictly increasing on $(0, \omega(F))$. Let the signature of the system be of the form $(0, ..., 0, s_k, s_{k+1}, ..., s_n)$, $s_k > 0$. Then the reliability function $\bar{F}_{T_{r:n}}(t)$ of $T_{r:n}$, $(1 \le r \le k)$, can be represented as follows,*

$$\bar{F}_{T_{r:n}}(t) = \exp\left\{ - \int_0^t \frac{1 + \frac{d}{dt} H^S_{r,n}(x)}{H^S_{r,n}(x) - \tilde{H}^S_{r,n}(x)} dx \right\}, \quad t \in (0, \omega(F)).$$

Let the signature be of the form $(s_1, s_2, ..., s_r, 0, ..., 0)$, $s_r > 0$. In a similar way as in Theorem 2, it can be verified that the distribution function $F_{T_{k:n}}(t)$ of $T_{k:n}$, $(r + 1 \le k \le n)$, can be represented as follows,

$$F_{T_{k:n}}(t) = \exp\left\{ - \int_t^{\omega(F)} \frac{1 - \frac{d}{dt} M^S_{k,n}(x)}{M^S_{k,n}(x) - \tilde{M}^S_{k,n}(x)} dx \right\}, \quad t \in (0, \omega(F)).$$

It should be pointed out here that $\tilde{H}^S_{r,n}(t) = H^{S'}_{1,n-r}(t) = E(T_{S'} - t \mid T_{1:n-r} > t)$, where $T_{S'}$ denotes the lifetime of the $(n - r)$-component *mixed* system $\mathcal{S}'$ with the signature vector $\mathbf{s}' = (0, ..., 0, s_k, s_{k+1}, ..., s_n)$. In other words, $\tilde{H}^S_{r,n}(t)$ is the MRL of the mixed system $\mathcal{S}'$, under the condition that all the components of the system are working at time t. Note that $\mathbf{s}'$ is a signature of order $(n - r)$. Similarly, if we consider the $(k - 1)$-component mixed system $\mathcal{S}''$ with lifetime $T_{S''}$ and the signature vector $(s_1, s_2, ..., s_r, 0, ..., 0)$, then $\tilde{M}^S_{k,n}(t) = E(t - T_{S''} \mid T_{k-1:k-1} \le t) = M^{S''}_{k-1,k-1}(t)$.

References

[1] Asadi, M. (2006). On the mean past lifetime of the components of a parallel system. *Journal of Statistical Planning and Inference*, **136**, 1197–1206.

[2] Asadi, M., and Bairamov, I. (2005). A note on the mean residual life function of the parallel systems. *Communications in Statistics: Theory and Methods*, **34**, 1–12.

[3] Asadi, M. and Bayramoglu, I. (2006). The mean residual life function of a k-out-of-n structure at the system level. *IEEE Transactions on Reliability*, **55** (2), 314–318.

[4] Barlow, R.E., Proschan, F. (1975). *Statistical theory of reliability and life testing*. Holt-Rinehart and Winston, New York.

[5] Boland, P.J. and Samaniego, F. (2004). The signature of a coherent system and its applications in reliability. In: Soyer R., Mazzuchi T., Singpurwalla, N. (eds.) *Mathematical reliability: an expository perspective*, international series in operational research and management science. Kluwer, Boston, pp. 1–29.

[6] Samaniego, F.J. (1985). On the closure of the IFR class under formation of coherent systems.*IEEE Transactions on Reliability*, R-**34**, 69–72.

[7] Tavangar, M. (2015). Conditional inactivity time of components in a coherent operating system. *IEEE Transactions on Reliability*, DOI: 10.1109/TR.2015.2422773.

[8] Tavangar, M. and Asadi, M. (2010). A study on the mean past lifetime of the components of $(n-k+1)$-out-of-n system at the system level. *Metrika*, **72**, 59–73.

[9] Tavangar, M. and Bairamov, I. (2015). On conditional residual lifetime and conditional inactivity time of k-out-of-n systems. *Reliability Engineering & System Safety*, **144**, 225–233.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

A note on the mixture representation of the Conditional Inactivity Time of a Coherent System

Teimoori, F. [1] and Goli, S. [2]

[1,2] Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran

Abstract

This article introduces a new mixture representation for the reliability function of conditional inactivity time of coherent system consisting of n independent and identically distributed components under a specific condition on the status of the system components. Also we drive some stochastic comparisons of the proposed conditional inactivity time.

Keywords: Coherent system, Inactivity time, Order statistics,Signature, Stochastic order.

1 Introduction

A coherent system is a technical structure consisting of no irrelevant component (a component is said to be irrelevant if its performance does not affect the performance of the system) and having a structure function that is monotone in each argument. Recently, the inactivity times of a coherent system or its components have been studied under different conditions; see, for example, Asadi (2006)[1], Zhang (2010)[5], Goliforushani and Asadi (2011)[2], Consider a coherent system comprising n components with i.i.d. lifetimes $X_1, X_2, ..., X_n$ distributed according to a common continuous distribution F. Suppose $T = T(X_1, X_2, ..., X_n)$ denotes the system lifetime. The concept of signature of coherent systems is a useful tool in the study of the reliability of coherent systems. In this paper, we consider a coherent system in which the signature vector is of the following form:

$$\mathbf{s} = (s_1, ..., s_i, 0, ..., 0), \tag{1.1}$$

[1]f.teimoori@math.iut.ac.ir
[2]s.goli@cc.iut.ac.ir

where $s_k > 0$ for $k = 1, 2, ..., i$, $i = 1, 2, ..., n-1$. A coherent system with the signature of the form (2.1) has the property that, upon the failure of the system at time t, components of the system with lifetimes $X_{k:n}, k = i+1, i+2, ..., n$, will be still alive. The study of the reliability properties of such a system may be of interest for engineers and system designers because after the failure of the system, the live components in the system can be removed and used for some other testing proposes. The rest of this paper is organized as follows. In Section 2, we introduce some stochastic orders and definitions which are used throughout the paper and we present a new mixture representation of the conditional inactivity time of coherent system in terms of conditional inactivity times of order statistics and then use them to obtain some stochastic ordering proporties of the conditional inactivity times.

2 Main results

Before proceeding to present the main results, we briefly introduce some stochastic orders for two nonnegative continuous random variables X and Y with respective distribution functions F and G , density functions f and g, and reliability functions $\overline{F}$ and $\overline{G}$.

Definition 1. *The random variable X is said to be less than the random variable Y in the*

i) stochastic order, denoted by $X \leq_{st} Y$, if $\overline{F}(x) \leq \overline{G}(x)$ for all $x > 0$;

ii) reversed hazard order, denoted by $X \leq_{rh} Y$, if $\frac{F(x)}{G(x)}$ is a decreasing function of x;

iii) likelihood ratio order, denoted by $X \leq_{lr} Y$, if $\frac{f(x)}{g(x)}$ is a decreasing function of x.

Definition 2. *If $X \leq_{rh} Y$,then*

$$P(X - t \leq -s | X \leq t) \geq P(Y - t \leq -s | Y \leq t), \quad \text{for all } s \geq 0 \text{ and all } t,$$

or equivalently

$$[X | X \leq t] \leq_{st} [Y | Y \leq t], \quad \text{for all } t.$$

We now our attention to studying the inactivity time of the coherent system that has failed by time t, i.e., $T < t$, but at time t, exactly l components have failed. In this case, we can define the following conditional random variables:

$$(t - T | T < X_{l:n} < t < X_{l+1:n}), \qquad l = i+1, i+2, ..., n-1.$$

The reliability function of this conditional random variable is given by

$$\begin{aligned} P(t - T > x | T < X_{l:n} < t < X_{l+1:n}) &= \sum_{m=1}^{l-1} P(T = X_{m:n}, t - T > x | T < X_{l:n} < t < X_{l+1:n}) \\ &= \sum_{m=1}^{l-1} p_{l,m}(t) P(t - X_{m:n} > x | X_{l:n} < t < X_{l+1:n}). \end{aligned}$$

We then have

$$P(t - T > x | T < X_{l:n} < t < X_{l+1:n}) = \sum_{m=1}^{l-1} p_{l,m}(t) C^X_{k,l,n}(t, x), \tag{2.2}$$

where

$$\begin{aligned} p_{l,m}(t) &= \frac{s_m P(X_{l:n} < t < X_{l+1:n})}{P(T < X_{l:n} < t < X_{l+1:n})} \\ &= \frac{s_m}{\sum_{u=1}^{l-1} s_u}, \qquad m = 1, ..., l-1 \\ &= p_m, \qquad m = 1, ..., l-1. \end{aligned}$$

It is obvious that $p_{l,m}(t)$ is independent of t and l, and so we simply have

$$p_{l,m}(t) = (p_1, ..., p_{l-1}, 0, 0, .., 0). \tag{2.3}$$

Let X and Y denote two continuous random variables with distribution functions F and G, density functions f and g, and reversed hazard rates r_F and r_G, respectively. Consider two coherent systems with the same signature as in (2.3), and let T_1 and T_2 denote the lifetimes of the systems whose components are distributed as F and G, respectively. We now prove that when the components of the system are ordered in terms of reversed hazard rates, then the inactivity times of the failed components of the systems are stochastically ordered. First, let $X_{i:n}$ and $Y_{i:n}$, $i = 1, 2, ..., n$, denote the ordered lifetimes of the components of the two systems, respectively. It can be easily shown that $X \leq_{rh} Y$ if and only if

$$(t - X_{m:n} | X_{l:n} < t < X_{l+1:n}) \geq_{st} (t - Y_{m:n} | Y_{l:n} < t < Y_{l+1:n}).$$

Theorem 1. *If $X \leq_{rh} Y$ and T_1 and T_2 denote the lifetimes of two systems with signature vectors as in* (2.3)*, then*

$$(t - T_1 | T_1 < X_{l:n} < t < X_{l+1:n}) \geq_{st} (t - T_2 | T_2 < Y_{l:n} < t < Y_{l+1:n}).$$

Proof. Note that

$$\begin{aligned} P(t - T_1 > x | T_1 < X_{l:n} < t < X_{l+1:n}) - P(t - T_2 > x | T_1 < Y_{l:n} < t < Y_{l+1:n}) \\ = \sum_{m=1}^{l-1} p_m (C^X_{k,l,n}(t, x) - C^Y_{k,l,n}(t, x)) \end{aligned}$$

From Remark 2, we have $C^X_{k,l,n}(t, x) \geq C^Y_{k,l,n}(t, x)$, and so

$$P(t - T_1 > x | T_1 < X_{l:n} < t < X_{l+1:n}) - P(t - T_2 > x | T_1 < Y_{l:n} < t < Y_{l+1:n}) \geq 0,$$

as required. □

Theorem 2. *Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the vectors of coefficients in* (2.3) *for two coherent systems of order n, both based on components with i.i.d. lifetimes distributed as the common continuous distribution function F. Let T_1 and T_2 be the corresponding lifetimes of the systems.*

(i) If $\mathbf{p}_1 \leq_{st} \mathbf{p}_2$, then $(t - T_1 | T_1 < X_{l:n} < t < X_{l+1:n}) \geq_{st} (t - T_2 | T_2 < Y_{l:n} < t < Y_{l+1:n})$;

(ii) If $\mathbf{p}_1 \leq_{rh} \mathbf{p}_2$, then $(t - T_1 | T_1 < X_{l:n} < t < X_{l+1:n}) \geq_{rh} (t - T_2 | T_2 < Y_{l:n} < t < Y_{l+1:n})$;

(iii) If $\mathbf{p}_1 \leq_{lr} \mathbf{p}_2$, then $(t - T_1 | T_1 < X_{l:n} < t < X_{l+1:n}) \geq_{lr} (t - T_2 | T_2 < Y_{l:n} < t < Y_{l+1:n})$.

Theorem 3. *If $r(t)$ is decreasing, then $P(t - T > x | X_{l:n} < t < X_{l+1:n})$ is increasing in t for all $x \geq 0$.*

References

[1] Asadi, M. (2006). On the mean past lifetime of components of a parallel system. *J. Stat. Plann. Inf.*, **136**, 1197-1206.

[2] Goliforushani, S. and Asadi, M. (2011). Stochastic ordering among inactivity times of coherent systems. *Sankhya, Series B,* **73**, 241-262.

[3] Khaledi, B.E. and Shaked, M. (2007). Ordering conditional lifetimes of coherent systems. *J. Stat. Plann. Inf.*, **137**, 1173-1184.

[4] Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.

[5] Zhang, Z. (2010). Mixture representations of inactivity times of conditional coherent systems and their applications. *J. Appl. Probab.,* **47**, 876-885.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

The New Burr Distribution with Application to Bladder Cancer Susceptibility

Tondpour, Z. [1] and Yari, GH. [2]

[1,2] School of Mathematics, Iran University of Science and Technology, Tehran, Iran

Abstract

This paper derives a new family of Burr type distributions as New Burr distribution. This particular skewed distribution can be used quite effectively in analyzing lifetime data. It is observed that the new distribution has modified unimodal hazard function. Estimation of parameters and change-point of hazard function by the maximum likelihood method are discussed. Change-point of hazard function is usually of great interest in medical or industrial applications. The flexibility of the new model is illustrated with an application to a real data set.

Keywords: Burr distributions, Change-point, Goodness-of-fit, Modified unimodal hazard function, Lifetime data analysis

1 Introduction

Irving Burr (1942) developed the system of Burr distributions. Analogous to the Pearson system of distributions, the Burr distributions are solutions to a differenial equation, which has the form

$$\frac{dy}{dx} = y(1-y)g(x,y), \tag{1.1}$$

where y equal to $F(x)$ and $g(x,y)$ must be positive for y in the unit interval and x in the support of $F(x)$. Different functional forms of $g(x,y)$ result in different solutions $F(x)$, which define the families of the Burr system.

[1]ztondpoor@mathdep.iust.ac.ir
[2]yari@iust.ac.ir

In this paper, we derives a New Burr distribution by replacing $g(x,y)$ with $g(x) = \frac{3px^2e^{-x^3}(1+e^{-x^3})^{p-1}}{(1+e^{-x^3})^p-1}$, $(p>0)$. If $g(x,y)$ is taken to be $g(x)$, then the solution of the differential equation (1.1) is given by:

$$F(x) = (e^{-G(x)}+1)^{-1}, \tag{1.2}$$

where $G(x) = \int g(x)dx$.
Then cdf and pdf of New Burr distribution are, respectively, given by:

$$F(x) = (1+e^{-x^3})^{-p}, \quad -\infty < x < \infty, \quad (p>0), \tag{1.3}$$

and

$$f(x) = 3px^2e^{-x^3}(1+e^{-x^3})^{-p-1}, \quad -\infty < x < \infty. \tag{1.4}$$

If the location parameter μ and the scale parameter σ are introduced in the equation (1.3), we have

$$F(x) = (1+e^{-(\frac{x-\mu}{\sigma})^3})^{-p}, \quad -\infty < x < \infty, \quad (p,\sigma>0,\ \mu\in\mathbb{R}) \tag{1.5}$$

and

$$f(x) = \frac{3p}{\sigma}(\frac{x-\mu}{\sigma})^2e^{-(\frac{x-\mu}{\sigma})^3}(1+e^{-(\frac{x-\mu}{\sigma})^3})^{-p-1}, \quad -\infty < x < \infty. \tag{1.6}$$

Hence equation (1.5) is three parameter New Burr distribution. Hazard function associated with the New Burr distribution is

$$h(x) = \frac{\frac{3p}{\sigma}(\frac{x-\mu}{\sigma})^2e^{-(\frac{x-\mu}{\sigma})^3}(1+e^{-(\frac{x-\mu}{\sigma})^3})^{-p-1}}{1-(1+e^{-(\frac{x-\mu}{\sigma})^3})^{-p}}. \tag{1.7}$$

The shapes of density and hazard functions of the New Burr distribution for different values of shape parameter p are illustrated in Figure 1.

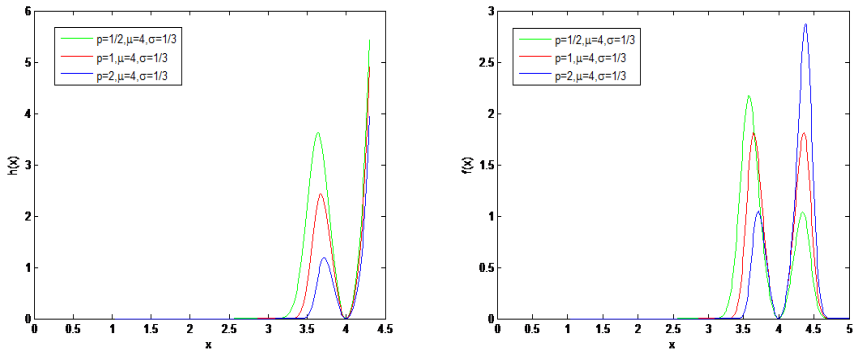


Figure 1: graphs of density and hazard functions of the New Burr distribution for different values of shape parameter p

2 Hazard change point estimation-classical approach

Hazard function plays an important role in reliability and survival analysis. New Burr distribution has modified unimodal (unimodal followed by increasing) hazard function. In some

medical situations, for example breast cancer, the hazard rate of death of breast cancer patients represents a modified unimodal shape.
A modified unimodal shape has three phases, first increasing, then decreasing, then again increasing. It can be interpreted as a description of three groups of patients, first group is represented by the first phase that contains the weak patients, so the hazard rate of this group is increasing, while the second phase represents the group with strong patients, their bodies have became familiar with the disease and they are getting better. The hazard rate of death of these patients is decreasing. In the third phase they become weaker and their ability to cope with the disease declines, then the hazard rate of death increases.
For situations where the hazard function is modified unimodal shaped, usually, we have interest in the estimation of lifetime change-point that is , the point at which the hazard function reaches to a maximum (minimum) and then decrease (increase). In reliability, the change-point of a hazard function is useful in assessing the hazard in the useful life phase. In this section, we consider maximum likelihood estimation procedure for change-point of the hazard function.
Let us assume that $x_1, ..., x_n$ is a random sample of size n of lifetimes generated by a New Burr distribution with parameters μ, σ and p. The log-likelihood function is given by:

$$l(\mu,\sigma,p) = n\log(\frac{3p}{\sigma}) + 2\sum_{i=1}^{n}\log(\frac{x_i-\mu}{\sigma}) - \sum_{i=1}^{n}(\frac{x_i-\mu}{\sigma})^3 -$$

$$(p+1)\sum_{i=1}^{n}\log(1+e^{-(\frac{x_i-\mu}{\sigma})^3}).$$

The maximum likelihood estimates for μ, σ and p denoted by $\hat{\mu}$, $\hat{\sigma}$ and $\hat{p}$, respectively, are obtained solving the likelihood equations, ($\frac{\partial l}{\partial \mu} = 0$, $\frac{\partial l}{\partial \sigma} = 0$ and $\frac{\partial l}{\partial p} = 0$).
From the invariance property of maximum likelihood estimators, we can obtain maximum likelihood estimators for functions of μ, σ and p. For $\phi = g(\mu,\sigma,p)$, a differentialable function of μ, σ and p, we have $\hat{\phi} = g(\hat{\mu},\hat{\sigma},\hat{p})$. Taking $\phi = h(x)$, defined in (1.7), the change-point of New Burr hazard function is obtained as solution of equation $\frac{d}{dx}\log(\phi) = 0$. The maximum likelihood estimator of the change-point is the solution of $\frac{d}{dx}\log(\phi) = 0$ with μ, σ and p replaced by maximum likelihood estimates. We observe that $\frac{d}{dx}\log(\phi) = 0$ is non-linear in x, so the change-point of the hazard function estimate should be obtained using some one dimensional root finding technique like Newton-Raphson.

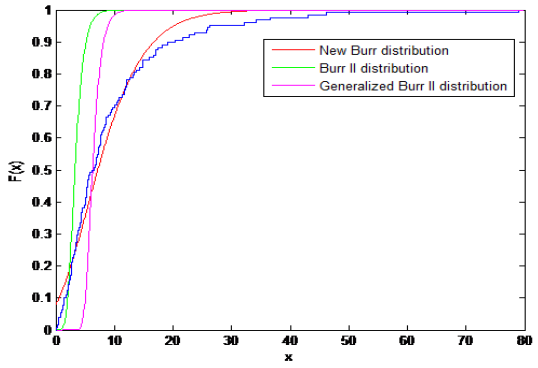


Figure 2: cdfs of the New Burr, Burr II and Generalized Burr II models for the remission times of bladder cancer data

3 Application

In this section, we use a real data set to compare the fits of the New Burr distribution and those of Burr II and Generalized Burr II. We consider an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients. These data were previously reported in Lee and Wang (2003). TTT plot for considered data is concave then convex indicating an increasing then decreasing failure rate function, is properly accommodated by New Burr distribution.

References

[1] Burr, I. W. (1942), Cumulative frequency distribution, *Annals of Mathematical Statistics*, **13**, 215-232.

[2] Lee, E. T. and Wang, J. W. (2003), *Statistical Methods for Survival Data Analysist*, 3rd edition, Wiley, New York.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Effect of imperfect ranking on estimator of stress-strength model based on ranked set sampling

Zamanzade, E. [1]

[1] Department of Statistics, University of Isfahan, Iran

Abstract

This paper is a short review on the estimation of stress-strength model based on ranked set sampling. We investigate the effect of imperfect ranking on the efficiency of ranked set sampling estimator with respect to simple random sampling estimator.

Keywords: Ranked set sampling, Imperfect ranking, Efficiency, Nonparametric estimator

1 Introduction

Suppose that X and Y be two independent random variables with distribution functions $F(x)$ and $G(y)$, respectively. One of the importan criterion in reliability is stress-strength model which is defined as $\theta = P(X > Y)$. The nonparametric estimation of θ based on two independent simple random sample $\{X_1, \ldots, X_{k_1}\}$ and $\{Y_1, \ldots, Y_{k_2}\}$ from two population with distribution $F(x)$ and $G(y)$, respectively, is given by

$$\hat{\theta}_{SRS} = \frac{1}{k_1 k_2} \sum_{i=1}^{k_1} \sum_{l=1}^{k_2} I(X_i > Y_l),$$

where $I(B)$ is the indicator function of set B.
One can easily verify that $\hat{\theta}_{SRS}$ is an unbiased estimator.
In the recent decades a new sampling design is taken into attention which is called ranked set sampling (RSS). RSS was originally proposed by McIntyre (1952) for increasing precision of mean estimator of pasture yields. This is a design for collecting data when actual measurement

[1] eli.zaman@sci.ui.ac.ir

of units are expensive, time consuming, but ranking small subsets of units are relatively easy and inexpensive. For drawing a ranked set sample with size $N = km$, first draw a simple random sample with size k^2 from population and divide it into k sets with size k. Rank units within each set from smallest to largest without actual (judgement ranking). For measuring, select 1th smallest unit from 1th set and 2th smallest unit from 2th set until draw kth smallest unit from kth set, these measured units construct a ranked set sample with size k and 1 cycle. Repeat this procedure for m times, a ranked set sample with size $N = km$ and m cycle will be obtained. This sample is indicated by $\{X_{[i]j}, i = 1, \ldots, k, j = 1, ., m\}$, where $X_{[i]j}$, ith judgement order statistics in jth cycle. Using (.) instead of [.] shows that the rankings are perfect which means that $\{X_{(i)j}, i = 1, \ldots, k, j = 1, \ldots, m\}$ are independent order statistics from samples with size k from population.
A ranking mechanism is called consistent if the following equation holds

$$F(x) = \frac{1}{k}\sum_{i=1}^{k} F_{[i]}(x), \ \forall x,$$

where $F(x)$ is parent distribution function and $F_{[i]}$ is distribution function of ith judgement order statistics.
Now consider estimation θ by two independent ranked set samples,
$\{X_{(i)j}, i = 1, \ldots, k_1, j = 1, \ldots, m\}$ and $\{Y_{(l)h}, l = 1, \ldots, k_2, h = 1, \ldots, m\}$. Then the unbiased estimator of θ based on two ranked set samples is given by

$$\hat{\theta}_{[RSS]} = \frac{1}{k_1 k_2}\sum_{j=1}^{m}\sum_{h=1}^{m}\sum_{i=1}^{k_1}\sum_{l=1}^{k_2} I(X_{[i]j} > Y_{[l]h}).$$

Sengupta and Mukhuti (2008) obtained the variance of $\hat{\theta}_{[RSS]}$ and $\hat{\theta}_{SRS}$ and compared them as the following theorem.

Theorem 1 (Sengupta 2008)**.** *Under the consistent ranking mechanism,* $Var(\hat{\theta}_{[RSS]}) \leq Var(\hat{\theta}_{SRS})$ *equality holds iff* $F_{[i]} = F, \forall i = 1, 2, \ldots, k_1$ *and* $G_{[j]} = G, \forall j = 1, 2, \ldots, k_2$*, i.e. either* $k_1 = k_2 = 1$ *or both the distributions* F *and* G *are degenerate.*

The relative efficiency of $\hat{\theta}_{[RSS]}$ with respect to $\hat{\theta}_{SRS}$ is defined as follows:

$$RE := RE(\hat{\theta}_{[RSS]}, \hat{\theta}_{SRS}) = \frac{Var(\hat{\theta}_{SRS})}{Var(\hat{\theta}_{[RSS]})}.$$

Theorem 1 implies that $RE(\hat{\theta}_{[RSS]}, \hat{\theta}_{SRS}) \geq 1$. In order to investigate the relative efficiency in more details, in the next section we discuss effect of ranking error on the RE.

2 Main results

For evaluating performance of imperfect ranking on RSS estimator, four different models of the judgement ranking are used:

(1) Fraction of random ranking (M_1): Consider the distribution function of ith judgement order statistic $F_{[i]}$ as a mixture of the distribution of ith order statistic $F_{(i)}$ and parent distribution function F: $F_{[i]} = \lambda F_{(i)} + (1 - \lambda)F, \quad \lambda \in [0, 1]$.

(2) Fraction of inverse rankings (M_2): Consider the distribution function of the ith judgement order statistic $F_{[i]}$ as mixture of perfect and perfectly wrong rankings: $F_{[i]} = \lambda F_{(i)} + (1 - \lambda)F_{(k-i+1)}, \quad \lambda \in [0, 1]$.

Table 1: Simulated RE for $Exp(\alpha)$ under different choices of $k_1, k_2, \lambda, \alpha$ and M_i, $i = 1, 2, 3, 4$.

(k_1, k_2)	α	$\lambda = 1$	$\lambda = 0.8$				$\lambda = 0.5$			
			M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4
(5,5)	3.5	2.850	1.706	1.291	2.419	1.482	1.137	1.000	2.014	1.129
	1	2.897	1.708	1.246	2.335	1.443	1.156	1.002	2.081	1.096
	0.8	2.769	1.721	1.310	2.407	1.414	1.172	1.012	2.042	1.105
	0.1	2.797	1.675	1.284	2.458	1.465	1.203	1.015	2.032	1.128
(5,10)	3.5	3.349	1.871	1.359	2.975	1.511	1.205	1.000	2.550	1.166
	1	3.363	1.787	1.322	3.033	1.527	1.201	1.022	2.568	1.161
	0.8	3.274	1.849	1.346	2.902	1.532	1.207	1.013	2.555	1.155
	0.1	3.385	1.819	1.374	2.973	1.523	1.232	1.001	2.466	1.142
(10,10)	3.5	5.233	2.110	1.484	4.772	1.678	1.272	1.038	4.089	1.174
	1	5.311	2.037	1.430	4.796	1.683	1.220	1.016	4.134	1.192
	0.8	5.123	2.107	1.396	4.712	1.643	1.256	1.016	4.128	1.197
	0.1	5.188	2.015	1.409	4.644	1.686	1.244	1.027	4.190	1.169

(3) Fraction of neighbours (M_3): $F_{[i]}$ is a mixture of $F_{(i)}$ and the distributions of the adjacent order statistics: $F_{[i]} = \frac{(1-\lambda)}{2} F_{(i-1)} + \lambda F_{(i)} + \frac{(1-\lambda)}{2} F_{(i+1)}$, where $F_{(0)} := F_{(1)}$ and $F_{(k+1)} := F_{(k)}$.

(4) Concomitant (M_4): Rank variable of interest X according to values of concomitant variable W which has correlation λ with X.

These models were used by Frey et al. (2007) and Zamanzade et al. (2014).
In the simulation study, we considered parent distribution as $Exp(\alpha)$ and $Normal(0, \alpha^2)$, λ values as 1, 0.8, 0.5 and samples sizes (k_1, k_2) as $(5, 5), (5, 10), (10, 10)$. We are generating 10,000 samples for different parent distributions, different judgement ranking models, different sample sizes and different values of λ. For each combination a relative efficiency value was computed. Table 1 and 2 exhibit the results for different parent distributions. While RE is always greater than 1 but, the quantity can differ for different values of k_1, k_2, λ and judgement ranking models. The relative efficiencies are highest when $\lambda = 1$ (prefect ranking) and they decrease as the λ values decrease. For judgement ranking models the relative efficiency is highest when judgement ranking model is fraction of neighbours (M_3) and $\lambda = 0.8$. As the sample sizes increase relative efficiencies increase. Result for different parent distributions are not much remarkable. The efficiency was not affected by cycle variation, so results in the tables are just for one cycle ($m = 1$).

References

[1] Frey, J., Ozturk, O. and Deshpande, J. (2007), Nonparametric tests for perfect judgment rankings, *Journal of Statistical Planning and Inference*, **138**, 708–717.

[2] McIntyre, G. A. (1952), Method for unbiased selective sampling using ranked sets, *Australian Journal of Agricultural Research*, **3**, 385–390.

[3] Sengupta, S. and Mukhuti, S. (2008), Unbiased estimation of $P(X > Y)$ using ranked set sample data, *A Journal of Theoretical and Applied Statistics*, **42**, 223–230.

Table 2: Simulated RE for $N(0, \alpha^2)$ under different choices of $k_1, k_2, \lambda, \alpha$ and $M_i, i = 1, 2, 3, 4$.

(k_1, k_2)	α	$\lambda = 1$	$\lambda = 0.8$				$\lambda = 0.5$			
			M_1	M_2	M_3	M_4	M_1	M_2	M_3	M_4
(5,5)	3.5	2.810	1.659	1.283	2.414	1.589	1.215	0.999	2.096	1.175
	1	2.753	1.738	1.321	2.443	1.673	1.217	1.033	1.957	1.167
	0.8	2.748	1.712	1.320	2.416	1.613	1.150	1.022	2.009	1.153
	0.1	2.759	1.644	1.281	2.488	1.643	1.188	1.013	2.036	1.179
(5,10)	3.5	3.444	1.822	1.359	2.917	1.718	1.218	1.010	2.483	1.161
	1	3.458	1.808	1.347	2.997	1.745	1.224	1.023	2.558	1.175
	0.8	3.445	1.759	1.351	2.965	1.766	1.209	1.008	2.526	1.156
	0.1	3.415	1.850	1.412	2.880	1.746	1.224	1.043	2.561	1.175
(10,10)	3.5	5.115	2.010	1.451	4.685	1.999	1.251	1.028	4.086	1.186
	1	5.207	2.037	1.413	4.899	2.003	1.272	1.006	4.293	1.258
	0.8	5.164	2.089	1.447	4.736	2.001	1.286	1.023	4.069	1.224
	0.1	4.966	2.100	1.402	4.676	1.947	1.252	1.019	4.288	1.211

[4] Zamanzade, E. , Arghami, N. and Vock, M. (2014), A parametric test of perfect ranking in balanced ranked set sampling, *Communications in Statistics - Theory and Methods*, **43**, 4589–4611.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Preventive Maintenance of a Network Subject to External Shocks

Zarezadeh, S. [1] and Ashrafi, S. [2]

[1] Department of Statistics, Shiraz University, Shiraz 71454, Iran
[2] Department of Statistics, University of Isfahan, Isfahan 81744, Iran

Abstract

In this paper, we consider a network with absolutely reliable nodes whose links are subject to shocks based on a counting process. Each shock may lead to links failure and the network finally fails due to one of these shocks. Two age-based and shock-based approaches are suggested for policy of preventive maintenance of such networks. The results can be illustrated by considering some networks.

Keywords: Reliability, t-signature, counting process, signature.

1 Introduction

Preventive maintenance (PM) is an interesting problem in reliability engineering for improving the network (or system) performance. Let a new unit have lifetime X with cumulative distribution function (c.d.f.) F and start operating at $t = 0$. PM means that the unit is replaced by a new one with the same c.d.f. F at each of the time instants T, $2T$, $3T, \ldots$. Further, the unit is replaced by a new one at the time of unit failure which termed emergency repair (ER). It is assumed that the required time for the replacement of the unit is negligible. In the literature on PM, we are usually interested to optimum the maintenance parameters such as the maintenance period. To this, we first obtain some expressions for the mean incurred costs when the unit is working, e.g., costs per unit of time or costs for the maintenance period.

The PM problems are frequently studied for the single unit systems or simple multi-unit networks such as series networks. Recently, some researchers have explored the PM policy for the

[1] s.zarezadeh@shirazu.ac.ir
[2] s.ashrafi@sci.ui.ac.ir

network with more complex structures to obtain the corresponding optimal solutions; see [1], [2].

Consider a network with lifetime T having two states: *up*, and *down*. The network is subject to shocks which appear according to a counting process $\{\xi(t), t > 0\}$. Let each shock may lead to link failure and the network finally fail by one of these shocks. Further, it is assumed that W_i, $i = 1, \ldots, n$, denotes the number of links that fail at the i-th shock and $W_0 \equiv 0$, by convention. If $N(t)$ denotes the number of links that fail up to time t, then $N(t)$ takes values on $\{1, 2, \ldots, n\}$ and $N(t) = \sum_{i=0}^{\xi(t)} W_i$. Under the assumption that the process of shocks occurrence is independent of the number of failed links, Zarezadeh et al. [5] showed that

$$P(T > t) = \sum_{i=1}^{n} s_i^{\tau} P(N(t) \leq i - 1),$$

where s_i^{τ} is the i-th element of vector $\mathbf{s}^{\tau} = (s_1^{\tau}, \ldots, s_n^{\tau})$ which is called *t-signature*. This vector depends only on the network structure and is a variant of the concept of *signature* defined by [4] in which avoids the restriction of not allowing the ties. With $\bar{S}_j^{\tau} = \sum_{i=j+1}^{n} s_i^{\tau}$, it was shown that

$$P(T > t) = \sum_{k=0}^{\infty} \beta_{k,n} P(\xi(t) = k), \tag{1.1}$$

where, for $k = 0, 1, \ldots$, $\beta_{k,n} = \sum_{j=0}^{n-1} \bar{S}_j^{\tau} P(\sum_{i=0}^{k} W_i = j)$.

Let $\vartheta_1, \vartheta_2, \ldots$ be the epoch times corresponding to $\{\xi(t), t > 0\}$. Then $\beta_{k,n} = P(T > \vartheta_k)$, and as a function of k, $\beta_{k,n}$ is a survival function with probability mass function $\mathbf{b}_n = (b_{1,n}, b_{2,n}, \ldots)$ where $b_{k,n} = P(T = \vartheta_k)$, $k = 1, 2, \ldots$.

In this paper, we consider the model described above and the PM on the network based on relation (1.1) is investigated. We use two approaches for this purpose: age-based PM and shock-based PM. The age-based approach is more traditional and the PM parameters are determined based on the time of the network failure. The latter is a novel approach in which the number of shocks is considered for obtaining optimal PM actions. Due to the limitation on the pages number of the paper, the illustrative examples are removed.

2 Age-based preventive maintenance

Let c_0 denote the cost of replacement of a failed link with a new one and c_{ER} be the cost of the emergency repair (ER). Based on the model described in (1.1), the expected cost of the renewal of the network is

$$C = c_0 \sum_{i=1}^{\infty} \beta_{i-1,n} E(W_i) + c_{ER},$$

where $\beta_{0,n} = 1$. If c_{PM} is the cost which is incurred by PM, the cost of the PM of the working network is equal to $c_0 \sum_{i=1}^{k} E(W_i) + c_{PM}$ when the PM is applied at the k-th shock. It is clear that $c_{PM} < c_{ER}$. Assume that the network is replaced at T_{PM} which is the failure time or the last renewal point. Then the expected cost for one cycle can be expressed as

$$\begin{aligned} S(T_{PM}) = &\sum_{i=0}^{\infty} P\big(\xi(T_{PM}) = i\big) \beta_{i,n} \big(c_0 \sum_{j=1}^{i} E(W_j) + c_{PM}\big) \\ &+ \sum_{i=1}^{\infty} P\big(\xi(T_{PM}) = i\big) \big(c_0 \sum_{j=1}^{i} (E(W_j) \sum_{k=j}^{i} b_{k,n}) + c_{ER}(1 - \beta_{i,n})\big) \end{aligned}$$

It can be seen that the expectation of the minimum of a random variable X and real number c is $\int_0^c \bar{F}(x)dx$ where $\bar{F} = 1 - F$ is the survival function of X. Then the expected duration of the renewal cycle for the network with the PM and shocks is equal to $\int_0^{T_{PM}} \bar{F}_T(t)dt$ in which $\bar{F}_T$ is given in (1.1). From the renewal reward theorem (see, e.g., [3]), the long-run expected cost per unit of time is equal to the expected cost per unit of time for one renewal cycle. Then the expected cost per unit of time is

$$C(T_{PM}) = \frac{S(T_{PM})}{\int_0^{T_{PM}} \bar{F}_T(t)dt}$$

and hence we need to obtain $C^*_{T_{PM}} = \inf\{C(T_{PM}), T_{PM} \geq 0\}$.

3 Shock-based PM

In this section, to apply the PM policy, we consider the number of shocks but not the times of the shocks occurrence. The PM is performed immediately after the k-th shock or the ER, whichever comes first. The expected number of shocks before the network failure is computed using the following relation

$$L(k) = \sum_{i=1}^{k} i b_{i,n} + k\beta_{k,n}.$$

It can be shown that the mean cost per cycle is

$$R(k) = \beta_{k,n}\Big(c_0 \sum_{j=1}^{k} E(W_j) + c_{PM}\Big) + c_0 \sum_{j=1}^{k} \Big(E(W_j) \sum_{i=j}^{k} b_{i,n}\Big) + c_{ER}(1 - \beta_{k,n})$$

Then we apply PM when the $k_{\min}$-th shock occurs so that

$$k_{\min} = \inf\{D(k), k = 1, 2, \dots\}, \qquad D(k) = \frac{R(k)}{L(k)}.$$

Therefore, we want to find an optimal $k_{\min}$ for the PM replacing all failed links. Obviously, in the case where $D(k)$ is strictly decreasing, no PM should be scheduled.

Let a network have n links with reliable nodes. Assume that each link of the network fails with probability p when a shock arrives and the links fail independent of each other. If the number of failed links in the first shock, W_1, has binomial distribution with parameters (n, p) and the number of failed links in the i-th shock, W_i, $i \geq 2$ has binomial distribution with parameters (n_i, p) where $n_i = n - \sum_{j=1}^{i-1} W_j$. With $q = 1 - p$, it can be seen that, from Lemma 2 of [5], $E\big(\sum_{i=1}^{k} W_i\big) = n(1 - q^k)$ and $E(W_k) = npq^{k-1}$, $k = 1, 2, \dots$. For such network with known t-signature, we can use the models described in Sections 2 and 3 for optimizing the PM parameters.

References

[1] Finkelstein, M. and Gertsbakh, I. (2015), 'Time-free' preventive maintenance of systems with structures described by signatures, *Applied Stochastic Models in Business and Industry*, **31**(6), 836-845.

[2] Finkelstein, M. and Gertsbakh, I. (2016), Preventive maintenance of multistate systems subject to shocks, *Applied Stochastic Models in Business and Industry*.

[3] Ross S.M. (1996), *Stochastic Processes*, 2nd edn, John Wiley and Sons, Inc.: USA.

[4] Samaniego, F.J. (1985), On closure of the IFR class under formation of coherent systems, *Reliability, IEEE Transactions on*, **34**(1), 69-72.

[5] Zarezadeh, S., Ashrafi, S. and Asadi, M. (2016), A shock model based approach to network reliability, *Reliability, IEEE Transactions on*.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Network Reliability Design Using Ranking and Selection Procedures

Mohammadi, Z. [1]

[1] Department of Statistics, Jahrom University, Jahrom 74135-111,, Iran

Abstract

This study was carried out to evaluate the problem associated with determining the most reliable network configuration. The proposed approach focuses on improving the efficiency of the evaluation of system reliabilities as well as quantifying the probability of correctly selecting the true best design based on the estimation of the expected system reliabilities through the use of ranking and selection procedures.

Keywords: Network Reliability Design, Ranking and Selection procedure, Probability of Correct Selection.

1 Introduction

Networ Reliability Design (NRD) is the difficult optimization problem associated with finding the topological configuration of an unreliable network, given certain design constraints, that maximizes the system reliability function. This problem has received considerable attention by researchers. Yang and Kubat [5] have proposed using theoretical bounds on the system reliability as a substitute for the actual reliability measure. As such, system reliability is typically approximated through simulation and estimation techniques such as Crude Monte Carlo simulation or more recently artificial neural networks [4]. For Crude Monte Carlo simulation, various approaches have been proposed to improve the efficiency of the simulation through different types of sampling techniques [1]. Ranking and Selection (R&S) is a group of statistical techniques used to justify the selection of the best or set of the best alternatives from a finite set of alternatives based on the estimation of their expected performance, Fu [2] provides reviews concerning how R&S is used throughout simulation optimization. In this study, we propose a ranking and selection method for the network reliability design problem.

In section 2, we formulate the NRD optimization problem and describe how Monte Carlo simulation is applied to solve it. Section 3, introduces the proposed method.

[1] z.mohammadi@jahrom.ac.ir

2 Formulation

let $G(V, E, T)$ represent a network with reliable node set $V = \{1, ..., n\}$, unreliable edge set $E = \{1, ..., m\}$, and terminal node set $T \subseteq V$ which represents the set of nodes that must be able to communicate with one another in order for the network to be considered operational. Now, we define the network reliability design optimization problem as follows,

$$\max_{x_i \in \theta} R(\mathbf{x}_i) \tag{2.1}$$

where θ is the set of feasible solutions defined by the following constraints:

$$\sum_{e=1}^{m} x_{ie} c_e \leq C_{max}$$

$$0 \leq x_{ie} \leq 1, e \in E, i$$

$$x_{ie} \in \mathbb{Z}^+ \cup \{0\}, e \in E$$

where c_e represents the individual cost of edge e and $\mathbf{x}_i = (x_{i1}, ..., x_{im})$ is the edge purchase vector of design i, where x_{ie} is the binary state of edge $e \in E$, such that

$$x_{ie} = \begin{cases} 1 & if \quad edge\ e\ is\ purchased\ in\ design\ i \\ 0 & if \quad otherwise \end{cases}$$

$R(\mathbf{x}_i)$ is the evaluation of the system reliability of design i, and C_{max} is the maximum allowable budget for purchasing edges in each design. Also, we assume θ is nonempty and finite, containing a total of K feasible designs, such that $K \leq 1000$. We can index the set of all candidate designs via $\theta = \{\mathbf{x}_1, ..., \mathbf{x}_K\}$, and represent the optimal solution(s) to (2.1) as $\theta^* = \{i : R_i > R_j, \forall j = 1, ..., K\}$ where $R_i \equiv R(\mathbf{x}_i)$. System reliability for this problem is defined as the probability that a given set of terminal nodes $T \subseteq V$ are connected at any given time, where the system is connected if all nodes in T can communicate with one another via operational edges. However, because the evaluation of the system reliability is difficult and cannot typically be solved in polynomial time for most networks, (2.1) is typically represented via:

$$\max_{x_i \in \theta} \widehat{R}(\mathbf{x}_i) \tag{2.2}$$

where $\widehat{R}(\mathbf{x}_i)$ is the approximation of R_i , based on Monte Carlo simulation. Solutions selected via (2.2) are just assumed to be optimal. Therefore, in order to achieve this purpose we propose the probability of correct selection (PCS) is at least $1 - \alpha$ i.e

$$p(select \widehat{R}_k \,|R_k - R_i \geq \delta \ \forall i \neq k \,\} \geq 1 - \alpha \tag{2.3}$$

where α is the allowable type I family-wise error and δ is the smallest difference worth detecting between any two system reliabilities, such that any two system reliabilities that differ by less than δ are considered equivalent. In order to implement (2.3) in terms of the NRD problem, we need to find a selection rule. Based on the published results in the ranking and selection literature, the procedure KN which is used in this study, has been shown to maintain superior overall performance as compared to the other procedures, for optimization problems similar to the NRD problem.

3 Procedure KN

In this section, we present the procedure KN. We define our samples in terms of the corresponding batch mean, such that

$$d_{ij} = \frac{1}{B} \sum_{l=1}^{B} \varphi(z_{ij}) \tag{3.4}$$

where d_{ij} represents the jth batch mean of the ith topological design of network G, and B is the number of i.i.d. Monte Carlo samples used in each batch. We then define the system reliability estimate for design i, as $\widehat{R}_i = \frac{1}{b_i}\sum_{j=1}^{b_i} d_{ij}$ where $b_i = \frac{n_i}{B}\ \forall i \in \{1, ..., K\}$ and n_i is the number of Monte Carlo samples.

Now, We follow this with a discussion of the individual details involved in each step.

1. Using (3.4) take b_0 i.i.d. batch means d_{ij} from each design i; set $b_{count} = b_0$, where b_0 is the batch mean counter variable for Procedure KN. And determine the system reliability estimates for all K designs based on their initial b_0 batch means, $\widehat{R}_i(b_0)\ \forall i \in \{1, ..., K\}$.

2. Determine the sample variance of the difference of the system reliability estimates of design i and design j determined in Step 1, as follows

$$S_{ij}^2 = \frac{1}{b_0 - 1}\sum_{l=1}^{b_0}(d_{il} - d_{jl} - (\widehat{R}_i(b_0) - \widehat{R}_j(b_0)))^2\ \forall i, j \in \{1, ..., K\}$$

3. Calculate $h = 2c\eta(b_0 - 1)$ where η is the solution to $g(\eta) = \beta$ represented by $\eta_c = \frac{1}{2}((\frac{2\alpha}{N-1})^{\frac{-2}{b_0-1}} - 1)$ if common random numbers are used and $\eta_l = \frac{1}{2}(2(1-(1-\alpha)^{\frac{1}{N-1}})^{\frac{-2}{b_0-1}} - 1)$ if independent replications are used, α is the user-specified allowable type I error, and c is a constant with recommended value $c = 1$.

4. Compute $b_{ij} = \left\lfloor (\frac{hS_{ij}}{\delta})^2 \right\rfloor\ \forall i, j \in \{1, ..., K\}$.

5. Set $b_j = \max_{j \neq i}\{b_0, b_{ij}\}\ \forall i$ and $b = \max_i b_i$.

6. Initialize $I = \{1, ..., K\}$, where I is defined as the set of candidate designs still in contention for the best.

7. Set $I^{old} = I$ and $I = (i : i \in I^{old}\ and\ \widehat{R}_i \geq \widehat{R}_l - W_{il}, \forall l \in I^{old}, l \neq i)$ where $W_{il} = \max(0, (\frac{\delta}{2cb_{count}})((\frac{hS_{ij}}{\delta})^2 - b_{count}))$.

8. If $|I| = 1$ stop and select the candidate design in I as the best.

Else if $b_{count} = b$, stop and select the design in I with the largest system reliability estimate as the best design. Else, take Δb additional batch means for all $i \in I$, set $b_{count} = b_{count} + \Delta b$, where Δb is the user-specified incremental increase for b_{count} between subsequent iterations, such that $1 \leq \Delta b$; set $\widehat{R}_i = \widehat{R}_i(b_{count})$ for all $i \in I$ and return to Step 7.

This process then continues to iterate in this manner until one of the stopping criteria are reached. It sould be noted that we had to omit the numerical examples of the proposed procedure provided.

References

[1] Fishman, G. S. (1986), A comparison of four Monte Carlo methods for estimating the probability of s-t connectedness. *IEEE Transactions on Reliability*, **35**(2), 145-155.

[2] Fu, M. C. (1994), A tutorial review of techniques for simulation optimization. *In Proceedings of the 26th conference on Winter simulation*. Society for Computer Simulation International: Orlando, Florida, United States.

[3] Kiekhaefer, A. P. (2011), Simulation ranking and selection procedures and applications in network reliability design. *Ph.D. Theses*, Iwoa university.

[4] Srivaree-ratana, C., Konak, A. and Smith, A. E. (2002), Estimation of all-terminal network reliability using an artificial neural network. *Computers and Operations Research*, **29**(7), 849-868.

[5] Yang, C. L. and Kubat, P. (1989), Efficient computation of most probably states for communication networks with multimode components. *IEEE Transactions on Communications*, **37**(5), 535-538.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Availability Optimization of Redundancy Allocation Problem for a System with Repairable and Non-Repairable Components

Zoulfaghari, H. [1] and Ahmadi, A. [2] and Zeinal Hamadani, A. [3]

[2] Department of Industrial Engineering, IHU University, Tehran, Iran
[1,3] Department of Industrial and Systems Engineering, Isfahan University of Technology, Isfahan, Iran

Abstract

Redundancy allocation problem (RAP) consists of two general classes, reliability optimization and availability optimization problems. In availability and reliability classes it is assumed that the entire components are repairable and non-repairable, respectively. In this paper a Mixed Integer Nonlinear Programming (MINLP) model is presented for modeling the problem of availability optimization of a system using repairable and non-repairable components, simultaneously. Also, it is shown that using different types of components in subsystems will increase the availability of the system. For solution of the proposed model, a genetic algorithm, as an efficient meta-heuristic algorithm, is developed and implemented. Furthermore, in order to indicate the efficiency of the proposed solution method, a numerical example for a system consisting of both repairable and non-repairable components is presented and solved. The results show a better performance for the proposed genetic algorithm compared to the Improved Particle Swarm Optimization (IPSO) algorithm that proposed at Literature review.

Keywords: Availability/Reliability Optimization, Redundancy Allocation Problem, Repairable and Non-Repairable Components, Genetic Algorithm.

[1]h.zoulfahjari@in.iut.ac.ir
[2]arahmady@ihu.ac.ir
[3]hamadani@cc.iut.ac.ir

1 Introduction

Reliability optimization problem is an important issue in system design, and it often tries to achieve the highest reliability for a system subject to several constraints such as cost, weight, and volume, etc. High reliability design enables a system to work more safety and efciently. In the past few decades, many approaches have been successfully proposed to solve this troublesome problem. Nowadays, decreasing production costs and using systems that are capable of operating in sensitive conditions has become more important than before due to the systems complications and competitive conditions. This necessity has resulted in significance of the concepts such as reliability and availability. Reliability of a system (or a component) is the probability that it will adequately perform its specied purpose for a specified period of time under specified environmental conditions [10] and availability is defined as the probability that a system is in its intended functional condition and therefore capable of being used in a stated environment [2]. Availability/reliability optimization of a system is the determination of the optimal number of components according to their characteristics and the structure of the system to maximize the availability/reliability. Redundancy Allocation Problem (RAP), as a method of improving the system reliability, is a complicated optimization problem which can be introduced as the selection of optimized combination of components type and redundancy levels for each subsystem in order to meet different needs with respect to all design constraints [6]. There are generally two groups of problems in this area [4]. The first group devotes its discussion to determination of type and number of redundant components for each subsystem. In this case, it is supposed that characteristics of each component such as reliability, weight, cost, etc. are pre-determined and the goal is to find the type and number of components that must be used in each subsystem to maximize its reliability. The second group considers component reliability as a decision variable; and its cost, weight and other characteristics are considered as predetermined increasing functions of component reliability. In this paper, the concerned problem pertains to the first group and the main difference is the assumption that system involves both repairable and non-repairable components. To solve this problem, a new mixed integer non-linear programming model is introduced and solved by a Genetic Algorithm (GA). To show the capability of proposed GA, a modified problem from the literature is considered and the GA results are compared to the result of the Improved Particle Swarm Optimization (IPSO) algorithm that proposed at Literature review. Over the past decades a number of optimization techniques have been developed in reliability-redundancy allocation problems. These techniques can be classied as implicit enumeration, dynamic programming, branch and bound method, linear programming, Lagrangean multiplier method, heuristic methods and so on. Hikita et al. [7] presented an application of the surrogate-constraints algorithm for an optimal reliability assignment/redundant allocation problem. Hsieh [8] presented a simple linear programming approach to approximate the integer nonlinear programming problem. Chen [2] presented a heuristic approach to solve reliability allocation problems with considering the weight, volume and cost constraint and proposed some solutions for this production problem and compared them with previous studies. Wu et al. [17] presented an improved particle swarm optimization (IPSO) algorithm to solve reliability problems. Yeh and Hsieh [18] proposed a penalty guided Articial Bee Colony algorithm (ABC) to solve the reliability redundancy allocation problem (RAP). Okasha et al. [13] proposed a novel maintenance optimization approach that integrates the system reliability and redundancy as objectives in addition to the life-cycle cost (LCC) objective. Their approach was able to optimally and automatically select what maintenance actions are applied, when they are applied, and to which structural components they are applied. In contrast to reliability optimization, fewer researchers have studied availability allocation and optimization to nd out the optimal failure and repair rates for each component in a system for maximizing (or minimizing) the objectives. In most cases, the problem of availability allocation and optimization can be dened as a multi-objective optimization problem, which aims to maximize system availability and minimize system cost [5]. Such as: Levitin and Lisnianski [11] introduced a model in which the cost of designing the system is fixed and its purpose is to optimize system availability. Also, Zio and Bazzo [20] presented an analysis on level Diagrams

of Pareto Front for redundancy allocation problem. Their aims were to maximize system availability and minimize the cost and weight of the whole system. Chiang and Chen [3] proposed a new multi-objective genetic algorithm, namely simulated annealing based multi-objective genetic algorithm (saMOGA), to resolve the availability allocation and optimization problems of a repairable series-parallel system. Tan et al. [15] developed a particle swarm optimization algorithm combined with a differential evaluation to solve a reliability-redundancy allocation problem. Zoulfaghari et al. [21,22] resolve the redundancy allocation problem and the availability redundancy allocation problem for optimization with Non Sorting Genetic Algorithm II (NSGA II).Zhang and Chen [19] proposed a particle swarm optimization for solving the multiobjective reliability redundancy allocation in an interval environment. Soltani et al. [14] proposed a robust optimization approach is used to solve the redundancy allocation problem (RAP) in series-parallel systems with component mixing where uncertainty exists in components reliabilities. Teimori et al. [16] used an efficient memory-based electromagnetism-like mechanism for the RAP.

2 Problem formulation

The series-parallel system as one of the well-established systems is used to describe and demonstrate the proposed approaches [5] The common structure of a series-parallel system is illustrated in Figure 1. Without loss of generality, suppose that all components are different in each subsystem. In general, the series-parallel system is considered with two objectives of maximizing system availability and minimizing system cost. As it is mentioned in previous sections, in most of the studies on redundancy allocation problem, optimization of system reliability has been considered and it has been supposed that all components are non-repairable. In fact availability and maintainability of the components have given less attention. Furthermore, in some cases where system availability is considered, it is supposed that the system consists of only repairable components. While in real world there are a few systems that are designed by using only either repairable or non-repairable components. In fact the most complicated systems consist of both repairable and non-repairable subsystems, as an example one can refer to systems composed of electronic and mechanical sections such as automobile motor system, airplane system, production systems, etc., where the electronic sections consist of non-repairable components while the mechanical sections have repairable components. Therefore, we assume that some subsystems use non-repairable components while some others have repairable ones. In this case, since some components of systems are repairable, it is not possible to use the reliability formulation for objective function; therefore, modeling should be done in a way that system availability is considered as the objective function for maximizing.

2.1 Mathematical model

In this section, a bi-objective mathematical model which is designed for the problem is presented. The suggested model is as follows;

$$Maxf_1 = Max\left\{\prod_i Av_i = \prod_{i\in R}\left(1-\prod_{j=1}^{m_j}\left(1-R_{ij}\right)^{n_{ij}}\right)\times\prod_{i\in A}\left(1-\prod_{j=1}^{m_j}\left(1-A_{ij}\right)^{n_{ij}}\right)\right\} \tag{2.1}$$

$$Minf_2 = \sum_{\mathrm{i}=1}^{\mathrm{s}}\sum_{j=1}^{m_i}\mathrm{c_{ij}}\ n_{ij} \tag{2.2}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{s}}\sum_{j=1}^{m_i}\mathrm{w_{ij}}\ n_{ij} \le \mathrm{W} \tag{2.3}$$

$$\sum_{\mathrm{i}=1}^{\mathrm{s}} \sum_{j=1}^{m_i} \mathrm{v_{ij}}\ \mathrm{n}_{ij} \leq \mathrm{V} \tag{2.4}$$

$$\mathrm{p_i} \leq \sum_{j=1}^{m_i} \mathrm{n}_{ij} \leq \mathrm{N_i}\ \forall \mathrm{i} = 1, 2, \ldots, \mathrm{s} \tag{2.5}$$

$$\mathrm{n}_{ij} \in Z^{+} \qquad \begin{matrix} \forall \mathrm{i} = 1, 2, \ldots, \mathrm{s} \\ \forall j = 1, 2, 3, \ldots, m_i \end{matrix} \tag{2.6}$$

In this model equations (2.1) and (2.2) demonstrate the objective functions which are maximizing the system reliability and minimizing the overall cost of the system respectively. Constraints (2.3) and (2.4) ensure the available weight and volume constraints. Constraint (2.5) is related to the maximum and minimum number of permitted components in each subsystem and the constraints (2.6) denote the domain of the variables. Chern [2] proved that redundancy allocation problem in its simplest form of series system is a NP-hard problem. Therefore, in order to maximize the objective function of the proposed model, it is reasonable to use meta-heuristic methods.

3 Proposed genetic algorithm

However, genetic algorithm is applied generally to solve the models with one objective function. But, in this paper genetic algorithm is used to solve the suggested bi-objective model. In the first step, the -constrained method is used and by finding the optimized value for the second objective function (cost), this value is used in the problem constraints, then the one-objective problem is solved by genetic algorithm. The steps of proposed Genetic algorithm are follows:Chromosome definition, Fitness function, Initial population,Selection, Crossover, Mutation, Stopping criteria.

Description of the algorithm we refused due to the lack of space.

4 A numerical example

This part of the paper includes an example whose data is a combination of those applied in [21,9] . In this example, the series-parallel system includes 8 sub-systems where sub-systems 1 to 3 have non-repairable components while sub-systems 4 to 8 have repairable components. Maximum allowable weight and volume for the system are 350 and 450, respectively. Maximum and minimum numbers of allowable components in each sub-system have been considered as 1 and 5, respectively. Tables 1 to 4 include details of problem.

4.1 Case1: The identical components in each subsystem

In this subsection, it is assumed that for each subsystem only one type of component can be used. To solve this problem, the proposed genetic algorithm has been used. The designed GA has been coded by MATLAB software and has been run on a computer with 2G of RAM. In this algorithm, crossover rate and mutation rate are 0.85 and 0.15, respectively. Also, population size and maximum generation are 100 and 500, respectively. In order to show the capability of genetic algorithm, the problem has been also solved by Improved Particle Swarm Optimization (IPSO) algorithm that proposed by Wu et al.[18]. They demonstrate that IPSO is a very capable algorithm to solve redundancy allocation problems. Therefore, this algorithm is suitable for comparison. Also, for the IPSO, population size PS=100, maximal number of iterations K=500 and mutation

Table 1: Availability of components

sub	Tp.1	Tp.2	Tp.3	Tp.4	Tp.5	Tp.6	Tp.7	Tp.8	Tp.9
NR-Sub1	0.94	0.91	0.89	0.75	0.75	-	-	-	-
NR-Sub2	0.97	0.86	0.70	0.66	-	-	-	-	-
NR-Sub3	0.96	0.82	0.72	0.71	0.67	-	-	-	-
R-Sub4	0.98	0.977	0.982	0.978	0.983	0.92	0.984	-	-
R-Sub5	0.995	0.996	0.997	0.997	0.9987	-	-	-	-
R-Sub6	0.971	0.973	0.971	0.976	-	-	-	-	-
R-Sub7	0.977	0.978	0.978	0.983	0.983	0.981	0.971	0.983	0.977
R-Sub8	0.984	0.983	0.987	0.981	-	-	-	-	

Table 2: Cost of components

sub	Tp.1	Tp.2	Tp.3	Tp.4	Tp.5	Tp.6	Tp.7	Tp.8	Tp.9
NR-Sub1	9	6	6	3	2	-	-	-	-
NR-Sub2	12	3	2	2	-	-	-	-	-
NR-Sub3	10	6	4	3	2	-	-	-	-
R-Sub4	59	53.5	47	42	40	18	22	-	-
R-Sub5	20.5	18.9	9.1	5.6	4.2	-	-	-	-
R-Sub6	752.5	472	359	242	-	-	-	-	-
R-Sub7	18	16	15	12.1	10.2	9.6	7.1	4.9	4.4
R-Sub8	98.6	82.5	49	47.5	-	-	-	-	-

probability Pm=0.1. Figure 2 shows the convergence of the Objective function value in each generation. This solution belongs to one of 20 times executions for cost equal to 1400. The near-optimal solution (Objective Function Value=0.96975691) is achieved after approximately 480 generations. Also, these results show that availability of the system increases when the cost is increasing. Pareto front of results are illustrated in figure 3 by using the median of availability. This figure shows that in each value of cost, obtained availability by GA is better than obtained availability by IPSO. These results have demonstrated that the GA has strong convergence and stability than IPSO algorithm. Figure 4 shows, the average run time of the GA compared to that of the IPSO for all test examples.

4.2 Case2: Different components in each subsystem

In this subsection, it is assumed that in subsystem i there are mi types of redundant components. Two algorithms are executed 20 times for each value of cost changing from 375 to 800 (units of cost) and the results are given in table 6. Figure 5 shows the convergence of the Objective function value in each generation. This solution belongs to one of 20 times executions for cost equal to 400. The near-optimal solution (Objective Function Value=0.96723014) is achieved after approximately 390 generations. Also, these results show that availability of the system increases when the cost is increasing. Pareto front of results are illustrated in figure 6 by using the median of availability. This figure shows that in each value of cost, obtained availability by GA is better than obtained availability by IPSO. These results have demonstrated that the GA has strong convergence and stability than IPSO algorithm. Figure 7 shows, the average run time of the GA compared to that of the IPSO for all test examples. Comparing the results of tables 5 and 6, it is interesting to notice that using different components in subsystems will increase the system availability. For example, when value of cost in case.1 is 1000 units, availability of system is equal to 0.88362519, while in case.2, the obtained availability is equal to 0.99995482. In other words, when different components are used in a subsystem, the model has more flexibility for making optimal situations and problem is more flexible for solution.

Table 3: Weight of components

sub	Tp.1	Tp.2	Tp.3	Tp.4	Tp.5	Tp.6	Tp.7	Tp.8	Tp.9
NR-Sub1	12	8	6	9	10	-	-	-	-
NR-Sub2	7	10	5	6	-	-	-	-	-
NR-Sub3	8	11	4	6	6	-	-	-	-
R-Sub4	40	40	39	38	39	38	37	-	-
R-Sub5	32	28	25	26	30	-	-	-	-
R-Sub6	53	51	50	50	-	-	-	-	-
R-Sub7	30	28	29	28	30	31	28	30	20
R-Sub8	29	27	26	27	-	-	-	-	-

Table 4: Volume of components

sub	Tp.1	Tp.2	Tp.3	Tp.4	Tp.5	Tp.6	Tp.7	Tp.8	Tp.9
NR-Sub1	12	8	6	9	10	-	-	-	-
NR-Sub2	7	10	5	6	-	-	-	-	-
NR-Sub3	8	11	4	6	6	-	-	-	-
R-Sub4	40	40	39	38	39	38	37	-	-
R-Sub5	32	28	25	26	30	-	-	-	-
R-Sub6	53	51	50	50	-	-	-	-	-
R-Sub7	30	28	29	28	30	31	28	30	20
R-Sub8	29	27	26	27	-	-	-	-	-

5 Summary and Conclusions

Reliability and availability measures are very important characteristics in many systems especially electronic and mechanical systems. Providing secure and reliable systems require special attention to these features. In most of previous studies in this area, it is always supposed that systems either have just repairable components or they merely include non-repairable components. In this paper, it is supposed that the system has both repairable and non-repairable components, simultaneously. On the base of this assumption, a new mathematical model is introduced and solved by genetic algorithm. Results produced by genetic algorithm shows that the system availability has always been in an appropriate level and compared to other common methods, the precision and speed of the genetic algorithm is higher than other methods. Also, the results show that the availability of a system will increase if there are different components in subsystems. For future studies one can extend the model such a way that each subsystem could have different components in terms of cost, availability (or reliability), weight, volume, etc. by considering either redundancy allocation or availability allocation.

References

[1] Chen, T. C. (2006), IAs based approach for reliability redundancy allocation problems. *Applied Mathematics and Computation*, **182**(2), 1556-1567.

[2] Chern, M. S. (1992), On the computational complexity of reliability redundancy allocation in a series system. *Operations research letters*, **11**(5), 309-315.

[3] Chiang, C. H., Chern, L. H. (1996), Availability allocation and multi-objective optimization for parallelseries systems. *European journal of operational research*, **180**(3), 1230-1244.

Table 5: Results of solving the problem Case 1

Cost	Alg.	Time(s)	Best	Worst	Median	SD
1000	GA	0.059	0.89247585	0.87059235	0.88362519	2.693e-005
	IPSO	0.069	0.88972537	0.87062148	0.87295147	3.515e-005
1200	GA	0.071	0.96952367	0.96022741	0.96258412	6.982e-005
	IPSO	0.094	0.96942579	0.95993671	0.95229646	7.986e-005
1400	GA	0.077	0.97268536	0.96982305	0.97038265	2.015e-005
	IPSO	0.106	0.97263289	0.96895231	0.95872873	5.771e-005
1600	GA	0.083	0.97669899	0.97512258	0.97620587	7.059e-005
	IPSO	0.127	0.97610289	0.97328895	0.96452378	8.029e-005
1800	GA	0.094	0.98886527	0.98362715	0.98793627	3.298e-005
	IPSO	0.139	0.98693328	0.98299716	0.97582694	6.623e-005
2000	GA	0.104	0.99863527	0.99715874	0.99802517	4.395e-005
	IPSO	0.148	0.99820148	0.99002691	0.99262973	9.439e-005

[4] Coit, D. W., Smith, A. E. (2003), Reliability optimization of series-parallel systems using a genetic algorithm. *Operations research letters*, **11**(5), 309-315. , IEEE Transactions on, **45**(2), 254-260.

[5] Elegbede, C., Adjallah, K. (2003), Availability allocation to repairable systems with genetic algorithms: a multi-objective formulation. *Reliability Engineering System Safety*, **82**(3), 319-330.

[6] Hamadani, A. (1980), *Availability and Reliability Modeling*, Ph.D. dissertation, University of Bradford, England.

[7] Hikita, M., Nakagawa, Y., Nakashima, K. , Narihisa, H. (1992), Reliability optimization of systems by a surrogate-constraints algorithm. *Reliability, IEEE Transactions on*, **41**(3), 473-480.

[8] Hsieh, Y. C. (2003), A linear approximation for redundant reliability problems with multiple component choices. *Computers industrial engineering*, **44**(1), 91-103.

[9] Khalili-Damghani, K., Amiri, M. (2012), Solving binary-state multi-objective reliability redundancy allocation series-parallel problem using efficient epsilon-constraint, multi-start partial bound enumeration algorithm, and DEA. *Reliability Engineering System Safety*, **103**, 35-44.

[10] Kuo, W., Prasad, V. R. (2000), An annotated overview of system-reliability optimization. *Reliability, IEEE Transactions on*, **49(2)**, 176-187.

[11] Levitin, G., Lisnianski, A. (2001), A new approach to solving problems of multi-state system reliability optimization. *Quality and reliability engineering international*, **17(2)**, 93-104.

[12] Liang, Y. C., Chen, Y. C. (2007), Redundancy allocation of series-parallel systems using a variable neighborhood search algorithm. *Reliability Engineering System Safety*, **92(3)**, 323-331.

[13] Okasha, N. M., Frangopol, D. M. (2009), Lifetime-oriented multi-objective optimization of structural maintenance considering system reliability, redundancy and life-cycle cost using GA. *Structural Safety* , **31(6)**, 460-474.

[14] Soltani, R., Safari, J., Sadjadi, S. J. (2015), Robust counterpart optimization for the redundancy allocation problem in series-parallel systems with component mixing under uncertainty. *Applied Mathematics and Computation* , **271**, 80-88.

Table 6: Results of solving the problem Case 2

Cost	Alg.	Time(s)	Best	Worst	Median	SD
375	GA	9.631	0.93512257	0.92845217	0.93485715	4.328e-005
	IPSO	10.789	0.93497512	0.92751264	0.93395217	9.934e-005
400	GA	10.237	0.96531258	0.96125894	0.96501256	3.874e-005
	IPSO	10.573	0.96512387	0.96124873	0.96454821	9.921e-005
450	GA	10.891	0.97985214	0.96985412	0.97853214	4.921e-005
	IPSO	11.162	0.97925487	0.96541297	0.97295471	7.219e-005
500	GA	11.746	0.98884597	0.98158742	0.98874592	5.281e-005
	IPSO	12.011	0.98859214	0.97625847	0.98295419	1.187e-004
550	GA	12.973	0.99058423	0.98388126	0.99015487	2.897e-005
	IPSO	13.299	0.98795213	0.98312947	0.98652914	6.715e-005
600	GA	13.294	0.99139985	0.99025473	0.99092154	4.985e-005
	IPSO	13.661	0.99078532	0.98962547	0.99015876	9.217e-005
650	GA	13.994	0.99675891	0.99602849	0.99658412	5.218e-005
	IPSO	14.243	0.99621584	0.99082164	0.99458167	9.842e-005
700	GA	14.588	0.99901224	0.99885236	0.99900265	3.927e-005
	IPSO	15.022	0.99875921	0.99812365	0.99857361	1.157e-004
750	GA	15.638	0.99902954	0.99895481	0.99900741	3.816e-005
	IPSO	16.199	0.99874125	0.99795264	0.99849571	8.129e-005
800	GA	16.332	0.99997521	0.99990125	0.99995482	4.892e-005
	IPSO	16.887	0.99954781	0.99902674	0.99938519	6.981e-005

[15] Tan, Y., Tan, G. Z., Deng, S. G. (2013), Hybrid particle swarm optimization with differential evolution and chaotic local search to solve reliability-redundancy allocation problems. *Journal of Central South University* , **20**, 1572-1581.

[16] Teimouri, M., Zaretalab, A., Niaki, S. T. A. , Sharifi, M. (2016), An efficient memory-based electromagnetism-like mechanism for the redundancy allocation problem. *Applied Soft Computing*, **38)**, 423-436.

[17] Wu, P., Gao, L. Zou, D. , Li, S. (2011), An improved particle swarm optimization algorithm for reliability problems. *ISA transactions*, **50(1)**, 71-81.

[18] Yeh, W. C., Hsieh, T. J. (2011),Solving reliability redundancy allocation problems using an artificial bee colony algorithm. *Computers Operations Research*, **38(11)**, 1465-1473.

[19] Zhang, E., Chen, O. (2016),Multi-objective reliability redundancy allocation in an interval environment using particle swarm optimization. *Reliability Engineering System Safety*, **145**, 83-92.

[20] Zio, E., Bazzo, R. (2011),Level Diagrams analysis of Pareto Front for multi-objective system redundancy allocation. *Reliability Engineering System Safety*, **96(5)**, 569-580.

[21] Zoulfaghari, H., Hamadani, A. Z., Ardakan, M. A. (2014), Bi-objective redundancy allocation problem for a system with mixed repairable and non-repairable components. *ISA transactions* , **53(1)**, 17-24.

[22] Zoulfaghari, H., Hamadani, A. Z., Ardakan, M. A. (2015),). Multi-objective availability-redundancy allocation problem for a system with repairable and non-repairable components. *Decision Science Letters* , **4(3)**, 289-302.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Likelihood Based Inference on Progressive type-II hybrid-censored data For Burr Type III Distribution

Arabi Belaghi, R. [1] Valizadeh Gamchi, F. [2] Bevrani, H. [3]

[1,2,3] Department of Statistics, University of Tabriz, Tabriz, Iran

Abstract

In this article we discuss about the problem of estimate the Burr type III distribution parameters based on progressive type II hybrid censored data. Because the estimation of these parameters is complex and has not closed form. So we use the EM and stochastic EM algorithm (SEM) for estimating the parameters of this distribution which based on the missing data. Then, a Mont Carlo simulation study is conducted to assess the accuracy of proposed estimators. The results revealed that SEM and EM have better performances than MLE estimates based on Newton-Raphson method and parameter estimates are improved when the sample size is increasing.

Keywords: Burr type III distribution, Progressive type II hybrid censored data, EM algorithm, Stochastic EM algorithm, maximum-Likelihood estimator.

1 Introduction

In many lifetime and reliability studies, we face the test units are excluded before failure time it observed in lifetime test. In many cases, removal of the test unit, is deliberate and pre-designed to save time and cost of testing are done. Kundu and Joarder [1] discussed type-I progressive hybrid censoring scheme. Further Childs et al.[2] Proposed the type-II progressive hybrid censoring scheme. In summary, Type II Progressive Hybrid Censoring Scheme is as follows: consider n identical, independent units with distinct distribution are placed in a lifetime test. each random variable $X_{1:m:n}$, $X_{2:m:n}$,...,$X_{m:m:n}$ has identically distributed, with p.d.f $f(x;\theta)$ and c.d.f $F(x;\theta)$, where θ denotes the vector of parameters (α, β). The correct values of m $(m < n)$ is a random variable. Let $R_1, R_2, ..., R_m$ are fixed before the start of the experiment are called progressive

[1]

[2]Fatemeh.valizadeh70@gmail.com

[3]

censoring scheme with $R_j > 0$ and $\sum_{j=1}^{m} R_j + m = n$ is specified in experiment. Under the Type-II Progressive Hybrid Censoring scheme, at the time of the first failure $X_{1:m:n}$, R_1 of the $n-1$ surviving units are randomly withdrawn from the experiment. Also, at the time of the second failure $X_{2:m:n}$, R_2 of then $n - R_1 - 2$ surviving units are withdrawn. And so on so forth, finally at the time of the m-th failure, $X_{m:m:n}$ all $R_m = n - R_1 - R_2 - ... - R_m - 1$ surviving units are withdrawn from the experiment. Hence,$X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ are called progressive censored failure times. The type-II Progressive Hybrid Censoring scheme includes the termination of the life-test at T^* =max{ $X_{m:m:n}, T$ }if $X_{m:n} > T$ the experiment would terminate at the m-th failure, with the withdrawal of units occurring after each failure according to the pre-fixed progressive censoring scheme $(R_1, R_2, ..., R_m)$. Although, if $X_{m:n} < T$, then instead of terminating the experiment by removing all remaining R_m units after the m-th failure, the test would continue to observe failures without any further withdrawals up to time T. Let D denote the number of failures that occur before time T. In which d shows the observed value of D. Thus, in this case, $R_m = R_{m+1} + 1 = ... = R_D = 0$. In this case, the resulting failures times is indicated by $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}, X_{m+1:n}, X_{d:n}$. We denote the two cases as case I and case II, respectively :

case I :$X_{1:m:n} < X_{2:m:n} < ... < X_{m:m:n}$, $X_{m:m:n} > T$;

case II : $X_{1:m:n} < ... < X_{m:m:n} < X_{m+1:m:n} < ... < X_{D:n}$, $X_{m:m:n} < T$.

The likelihood function based on the observed data is as follow :

Case I : $\mathrm{L}(\theta) = \mathrm{c}_1 \Pi_{j=1}^{m}$ f$(\mathrm{X}_{j:m:n}; \theta)$ $[1 - F(X_{j:m:n}; \theta)]^{R_j}$,

Case II : $\mathrm{L}(\theta) = \mathrm{c}_2 \Pi_{j=1}^{m}$ f($\mathrm{X}_{j:m:n}; \theta)[1 - F(X_{j:m:n}; \theta)]^{R_j} \Pi_{i=m+1}^{L} f(X_{i:n}; \theta)[1 - F(T)]^{\acute{R}_D}$.

Where $c_1 = n(n - R_1 - 1)...(n - R_1 - R_2 - ... - R_{m-1} - m + 1)$, $c_2 = n(n - R_1 - 1)...(n - R_1 - R_2 - ... - R_{m-1} - D + 1)$, $D = m + 1, ..., n - \Sigma_{k=1 m-1} R_k$, $\acute{R}_D = n - D - \Sigma_{k=1}^{m-1} R_k, R_m = 0$ if $D \geq m$ and $L = m + 1, ..., n - \Sigma_{k=1}^{m-1} R_k - \acute{R}_D$.
The usual Mamximum likelihood estimators (MLEs) can be obtained by solving the above nonlinear system resulted to have a poor estimates when the initial values missspecified therefor, we use a very stable and strong method in the next section.
Burr III has many application in reliabilty and quality control. Burr [3] introduced twelve distributions.The Burr type III with two parameters α and β has c.d.f and p.d.f as follows:

$$F(x \mid \alpha, \beta) = \left(1 + x^{-\beta}\right)^{-\alpha}, x > 0, \alpha > 0, \beta > 0, \tag{1.1}$$

$$f(x \mid \alpha, \beta) = \beta \alpha x^{-\beta - 1} \left(1 + x^{-\beta}\right)^{-\alpha - 1}. \tag{1.2}$$

In this study we are aiming to drew likelihood baesd estimation of the parameters based on progressive hybrid censored data. The rest of the paper is as follows.
in section 2 we obtanid the estimators based on EM, SEM and Newton-Raphson (NR) methods. Section 3 contanes the simulation study and an illustrative example. This paper will be end up by summary and conclusion in section 4.

2 Parameter Estimation

2.1 Maximum-likelihood estimators

The usual Mamximum likelihood estimators (MLEs) can be obtained by solving the above nonlinear system resulted to have a poor estimates when the initial values missspecified therefor, we use a very stable and strong method in the next section.

2.2 EM algorithm

EM algorithms is used for parameter estimation when we have missing data and an iterative algorithm is done in two stages. To make inferences about the type II progressive hybrid censored data. The results for the Case I is stright forward so we neglected.
In case II, we denote $X = (X_{1:m:n}, ..., X_{m:m:n}, X_{m+1:n}, ..., X_{D:n})$ and $Z_j = (Z_{11}, ..., Z_{1R_j}, ..., Z_{m1}, ..., Z_{mR_j})$,$\acute{Z} = (\acute{Z}_1, \acute{Z}_2, ..., \acute{Z}_{\acute{R}_D})$ respectively denote the observed and censored data, Z_j,$\acute{Z}$ are as the missing data.This algorithm was introduced by Dempster, Laird and Rubin [6]. So all data will be show $W = (X; Z_j, \acute{Z})$. If we denote D to be the number of failures before time T, Log-likelihood function is based on complete data for Burr III distribution as follows:

$$L(W;\alpha,\beta) = nlog\beta + nlog\alpha + (-\beta-1)[\sum_{j=1}^{m} logx_j + \sum_{j=1}^{m}\sum_{l=1}^{R_j} z_{jl} + \sum_{j=m+1}^{D} logx_j + \sum_{i=1}^{\acute{R}_j} log\acute{z}_i] +$$
$$(-\alpha-1)[\sum_{j=1}^{m} log(1+x_j^{-\beta}) + \sum_{j=1}^{m}\sum_{l=1}^{R^j} log(1+z_{jl}^{-\beta}) + \sum_{j=m+1}^{D} log(1+x_j^{-\beta}) + \sum_{i=1}^{\acute{R}_D} log(1+\acute{z}_i)].$$

Then for E-step we have :

$$\mathrm{L(W;}\alpha,\beta) = nlog\beta + nlog\alpha + (-\beta-1)[\textstyle\sum_{j=1}^{m} logx_j + \sum_{j=m+1}^{D} logx_j]$$
$$+ (-\beta-1)[\textstyle\sum_{j=1}^{m}\sum_{l=1}^{R_j} E(logZ_{jl}|Z_{jl} > x_m) + \sum_{i=1}^{\acute{R}_N} E(log\acute{z}_i \mid \acute{z}_i > T)]$$
$$+ (-\alpha-1)[\textstyle\sum_{j=1}^{m} log(1+x_j^{-\beta}) + \sum_{j=m+1}^{D} log(1+x_j^{-\beta})]$$
$$+ (-\alpha-1)[\textstyle\sum_{j=1}^{m}\sum_{l=1}^{R_j} E(log(1+Z_{jl}^{-\beta})|Z_{jl} > x_m) + \sum_{i=1}^{\acute{R}_D} E(log(1+\acute{z}_i^{-\beta}) \mid \acute{z}_i^{-\beta} > T)].$$

where

$$A_3(\alpha,\beta;T) = E(log\acute{Z}_i \mid \acute{Z}_i > T)$$
$$= \frac{\alpha\beta}{1-F(T;\theta)} \int_T^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1} logx dx$$

Also

$$A_4(\alpha,\beta;T) = E(log(1+\acute{Z}_i^{\beta}) \mid \acute{Z}_i > T)$$
$$= \frac{\alpha\beta}{1-F(T;\theta)} \int_T^{\infty} x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1} log(1+x^{-\beta}) dx$$

M-step
the estimators of (α,β) are (α^k,β^k), then $(\alpha^{k+1},\beta^{k+1})$ will be obtained by maximizing this func-

tion:

$$L(W;\alpha,\beta) = nlog\beta + nlog\alpha + (-\beta-1)[\sum_{j=1}^{m} logx_j + \sum_{j=m+1}^{D} logx_j]$$
$$+(-\beta-1)[\sum_{j=1}^{m} R_j A(\alpha,\beta;x_j) + \acute{R}_D A_3(\alpha,\beta;T)]$$
$$+(-\alpha-1)[\sum_{j=1}^{m} log(1+x_j^{-\beta}) + \sum_{j=m+1}^{D} log(1+x_j^{-\beta})]$$
$$+(-\alpha-1)[\sum_{j=1}^{m} R_j B(\alpha,\beta;x_j)) + \acute{R}_D A_4(\alpha,\beta;T)].$$

The fix point will be obtaine by repeating the following equations:

$$\hat{\alpha} = \frac{n}{\sum_{j=1}^{m-1} log(1+x_j^{-\beta}) + \sum_{j=m+1}^{D} log(1+x_j^{-\beta}) + \sum_{j=1}^{m} R_j B(\alpha,\beta;x_j) + \acute{R}_D A_4(\alpha,\beta;T)}.$$

$$\hat{\beta} = \frac{n}{[\sum_{j=1}^{m-1} logx_j + \sum_{j=m+1}^{D} logx_j] - (-\alpha-1)[\sum_{j=1}^{m-1} \frac{x_j^{-\beta} logx_j}{log(1+x_j^{-\beta})} + \sum_{j=m+1}^{D} \frac{x_j^{-\beta} logx_j}{log(1+x_j^{-\beta})}]}$$

$$\frac{n}{-\sum_{j=1}^{m} R_j A(\alpha,\beta;x_j) + \sum_{i=1}^{\acute{R_N}} A_3(\alpha,\beta;T)}$$

2.3 SEM algorithm

In some cases, the E-step in the EM algorithm has complex and unsolvable calculation. In this condition, this algorithm loses its efficiency. So in this case, the best solution is to use the stochastic EM algorithm, such as the Monte Carlo EM algorithm. Monte Carlo EM algorithm approximates the expectation in the e-step by the Monte Carlo Average (Zhang et al., 2013, p.712) [4]. Missing or unobserved or randomly-drawn-out censored data are replaced with a simulated sample of the conditional distribution of unobserved data according to the observed and complete data and in m-step log-likelihood of the complete data are maximized.

3 Simulation Study

In this part, simulation results to compare the performance of the maximum likelihood method. Then EM and Stochastic EM algorithm presented. For this purpose, Progressively Hybrid censored sample $n = 15, 40, 50$ the Burr III density function is generates. Then parameter distribution are estimated under different scheme censored. As well as estimators covariance be presented for diffrent n, m at the tables 1. Be considerate with increasing m estimated parameters are closer to the actual value. Simulation results it shows, in general the SEM algorithm estimators is better than EM and maximum likelihood estimator.

3.1 Simulation Results

In this section, a sample simulation results are presented for estimators in table 1. We have different values for n and m, and $T = 1.0, 1.5, 2.0$ for a selected scheme $R_j = (10, 0^{*19})$. We first generate type III Burr distribution based on type-II PHC Scheme data (see Gurulu Alama and Arabi Belaghi (2015)) [5], then parameter estimates are extracted under proposed methods with 1000 times simulation. According to the results, we observe that, Generally the SEM is supeiour to the EM which is outperform the Newton-Raphson.

Table 1: SEM, EM algorithm and Newton-Raphson estimation with covariance (scheme 2)

			SEM			EM			NR		
T	α	β	$\hat{\alpha}$	$\hat{\beta}$	$\text{cov}(\hat{\alpha},\hat{\beta})$	$\hat{\alpha}$	$\hat{\beta}$	$\text{cov}(\hat{\alpha},\hat{\beta})$	$\hat{\alpha}$	$\hat{\beta}$	$\text{cov}(\hat{\alpha},\hat{\beta})$
1.0	0.5	1	2.64	0.93	0.17	2.60	0.87	0.08	23234	11.68	-12442
1.0	0.5	2	2.56	1.92	0.38	2.71	0.41	-0.30	1741	12.14	1874
1.0	0.5	3	2.94	2.88	1.29	2.51	2.60	-0.14	3.02	8.40	3.05
1.5	0.5	1	2.53	0.97	0.209	2.61	0.86	0.07	3.68	12.53	-2.15
1.5	0.5	2	2.94	1.83	0.801	2.55	1.71	0.194	2.82	16.43	19.46
1.5	0.5	3	2.82	2.70	1.38	1.47	1.98	-1.54	0.75	0.52	10.86
2.5	0.5	1	2.53	0.95	0.205	2.61	0.84	0.079	3.37	12.35	-1.99
2.5	0.5	2	2.82	1.77	0.908	1.48	1.29	-1.03	1.51	12.40	13.77
2.5	0.5	3	2.84	2.34	0.803	1.51	1.81	1.674	0.24	7.82	1.93

3.2 Illustrative example and real-life Data Analysis

In this section, the suggested manner are applied for patients lifetime data with the white blood cells cancer in Rupert and Miller [7]. This data are shows 23 patients lifetime with white blood cells cancer,that is $33, 43, 45, 9, 13, 13, 18, 23, 28, 31, 34, 45, 48, 161, 5, 5, 8, 8, 13, 16, 23, 27, 30$. We fit Burr III for the data distribution and the Kolmogorov-Smirnov statistic are computed as KS = 0.15 resulted in the data has Burr III distribution. Then based on this data the estimator of the parameters are calculated and showen in table 2.

4 Summary and conclusion

In this paper we considered the estimation of the parameters of Burr III distribution under a so-called EM and SEM with progressive hybrid type two censored data. We then compared the results with those based on the usual maximum likelihood estimators via a Mont Carlo simulation study. The results showed that the proposed EM and SEM are superiour the usual MLEs that obtained from Newton-Naphson. This results can be compared with Bayes and emprical Bayes estimators but for the sake of space limitation we will do that in future studies.

References

[1] Kundu D, Joarder A. (2006), Analysis of type-II progressively hybrid censored data, *Comput Stat Data Anal*, **50**, 2509-2528.

[2] Childs A, Chandrasekar B, Balakrishnan N. (2008), Exact likelihood inference for an exponential parameter under progressive hybrid censoring, In: Vonta F, Nikulin M, Limnios N,

Table 2: SEM, EM algorithm and Newton-Raphson estimation

			SEM		EM		NR	
m	T	scheme	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
15	15	$(0^{*14}, 8)$	10.3717	0.6505	10.3701	0.6403	6.9027	0.0834
	25		10.2736	0.6467	10.2564	0.67862	6.9027	0.0834
	35		9.9619	0.6343	9.9238	0.6389	6.9027	0.0834
18	15	$(0^{*17}, 5)$	6.0292	0.4751	6.1667	0.4318	6.5874	0.0195
	25		5.9548	0.4704	5.9523	0.4629	6.5874	0.0195
	35		5.94017	0.4694	5.9156	0.4390	6.5874	0.0195
20	15	$(0^{*19}, 3)$	6.0292	0.4751	6.0496	0.4204	7.2296	0.6502
	25		5.9548	0.4704	5.9468	0.4829	7.2296	2.8288
	35		5.9401	0.4994	5.9406	0.4520	5.1668	4.1240

Huber-Carol C, editors. *Statistical models and methods for biomedicaland technical systems Boston, MA: Birkhuser*, p. 319330.

[3] Burr, I., W., (1942), Cumulative Frequency Functions, *The Annals of Mathematical Statistics* ,**1**, 251-232.

[4] Zhang.M,Ye.Zh,Xie.M,(2013) A Stochastic EM Algorithm for Progressively Censored Data Analysis, *Quality and Reliability Engineering International* , **30**, 711-722.

[5] Alma, G., O., Arabi Belaghi, R. (2015), On the estimation of the extreme value and normal distribution parameters based on progressive type-II hybrid-censored data, *Statistical Computation and Simulation*, **13**, 215-232.

[6] Dempster, AP., Laird, NM,. Rubin, DB. (1977), Maximum likelihood from incomplete data via the EM algorithm, *J R Statist Soc Ser B*, **39**(1), 1-38.

[7] Rupert, G., Miller, G., R. (1977) Survival Analysis,*John Wiley.*

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

An Algorithm for Computing the Multidimensional signature with Multithread Parallelization

Yarahmadi, A. [1] Zarezadeh, S. [2]

[1] Department of Computer Science, Engineering, and IT, School of Electrical and Computer Engineering, Shiraz University, Shiraz, Iran
[2] Department of Statistics, Shiraz University, Shiraz 71454, Iran

Abstract

Keywords: The multidimensional signature (D-spectrum) is a fundamental notion to explore the stochastic properties of multi-state networks (or systems). This concept depends only on the network structure and is a generalization of the signature which first defined by [2]. In this paper, we propose an algorithm to compute the multi-dimensional signature. We use parallelization method to reduce the run time of the algorithm.

Network reliability, multithreading, BFS algorithm.

1 Introduction

In many real situations, several states are considered for a network (or system), denoted by integer numbers $K = 0, 1, \ldots, M$. $K = M$ denotes the best state of the network in which the network has the complete performance (*up* state). $K = M-1, M-2, \ldots, 1$ correspond to the states of the network in the process of its gradual disintegration. The state $K = 0$ shows the complete failure of the network (*down* state). This type of networks is called multi-state networks. The reliability of the multi-state networks is an important problem for engineers and designers of the networks which has been investigated by numerous researchers.

From the point of view of mathematics, a network is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that includes a set of nodes (vertices), denoted by $\mathcal{V}$, and a set of links (edges), denoted by $\mathcal{E}$, that connect selected pairs of nodes. $\mathcal{T} \subseteq \mathcal{V}$ is considered as terminals set of the network and is used to indicate the network states. Let a multi-state network be in state $K = M$ at time $t = 0$. Assume that the links of the network are subject to failure and its nodes are absolutely reliable. It should be mentioned

[1] aryo_yarahmadi@yahoo.com
[2] s.zarezadeh@shirazu.ac.ir

that link failure means that the link is erased from the network. Let non-negative random variable T_1 denote the time that the network remains in state $K = M$. In the same way, random variable T_i denotes the time that the network enters state $K = M - i$, $i = 2, \ldots, M-1$ and by T_M the system lifetime, i.e. the time that the system goes into the state $K = 0$. Suppose $X_1, \ldots, X_n$ are independent and identically distributed random variables which denote the lifetimes of the network links. If $X_{1:n}, \ldots, X_{n:n}$ indicates the ordered lifetimes of the links, similar to (1.3.16) of [1], it can be shown that, for $t_1 > 0, \ldots, t_M > 0$

$$P(T_1 > t_1, \ldots, T_M > t_M) = \sum_{1 \le i_1 < \cdots < i_M \le n} \cdots \sum s_{i_1,\ldots,i_M} P(X_{i_1:n} > t_1, \ldots, X_{i_M:n} > t_M),$$

where $s_{i_1,\ldots,i_M} = P(T_1 = X_{i_1:n}, \ldots, T_M = X_{i_M:n})$; that is, the network enters states $K = M-1$, $K = M-2$, $\ldots$, $K = 0$ at the instant time of the i_1-th, i_2-th, $\ldots$, i_M-th link failure, respectively. The M-dimensional matrix $S = \{s_{i_1,\ldots,i_M}\}_{i_1,\ldots,i_M=1,\ldots,n}$ with size $n \times \cdots \times n$ is called the multidimensional signature. The matrix S depends only on the network structure and is free of the stochastic mechanism under which the links are failed. Gertsbakh and Shpungin [1] gave a combinatorial definition for the multidimensional signature. Suppose that $\pi = (e_{i_1}, e_{i_2}, \ldots, e_{i_n})$ denotes a permutation of the network link numbers. Let $n_{i_1,\ldots,i_M}$ be the number of permutations that the i_1-th, i_2-th, $\ldots$, i_M-th link failure cause the states of the network change from $K = M$ to $K = M-1$, from $K = M-1$ to $K = M-2$, $\ldots$, from $K = 1$ to $K = 0$, respectively. If all permutations are equally probable, the multidimensional signature matrix S has the non-zero elements as

$$s_{i_1,\ldots,i_M} = \frac{n_{i_1,\ldots,i_M}}{n!}, \qquad 1 \le i_1 < \cdots < i_M \le n.$$

In this paper, we propose an algorithm to compute the multi-dimensional signature. Multithreading is one method to execution of the algorithm which allows multiple threads to execute independently but share source code. We use the multithreading to speed up the execution time of the proposed algorithm. The performance of the algorithm is evaluated and simulation results are presented for some networks. Due to the limitation on the pages number of the paper, the experimental results are removed.

2 The Proposed Algorithm

Consider a network represented by a graph $\mathcal{G}$ in which the nodes are absolutely reliable and the links are subject to failure. Assume that some nodes of the network are considered as the terminals set which used to define the state of the network based on the connectivity between those. Assuming that all orders of the link failures are equally probable, in what follows, we propose a new algorithm to obtain the multidimensional signature matrix. In our algorithm, the adjacency matrix is used to determine the graph of the considered network. If the network has m nodes, the adjacency matrix has size $m \times m$ whose the (i, j)-th element is one if there is a link between the vertices i and j and is zero otherwise. To indicate the terminals of the network we let 3 on the i-th diagonal element of the matrix and 2 if the i-th node is not terminal. The adjacency matrix should be given by the user. Since the adjacency matrices that arise in applications contain only few nonzero elements, we apply *adjacency list* to reduce the used memory. The *make_adj_list* function converts adjacency matrix to adjacency list.

Let us first introduce the following notations.

Notations:

n	the number of network links
k	the number of network terminals
$\mathcal{T} = \{t_1, \ldots, t_k\}$	the terminals set of the network
$M+1$	the number of network states, $K = 0, 1, \ldots, M$
R	defines the rules to determine the states of the network
Adj_m	the adjacency matrix corresponding to graph $\mathcal{G}$
Adj_l	the adjacency list of graph $\mathcal{G}$
cluster	the number of clusters
state	the current state of the network
$\pi(r)$	the r-th element of vector π
N	the number of times that the algorithm is executed.
$N(r_1, \ldots, r_M)$	the number of times that the network state change to $K = M-1, \ldots, K = 0$ at the instant time of the r_1-th, $\ldots$,r_M-th link failure, respectively.

Algorithm

1. Initialize *Adj_m* matrix and load rules R
2. *Adj_l*$\leftarrow$*make_adj_list(Adj_m)*
3. $N(r_1, \ldots, r_M) \leftarrow 0, \quad r_1 = 1, ..., n, \ldots, r_M = 1, ..., n$
4. $\pi(1, \ldots, n) \leftarrow$*Perm*$(1, \ldots, n)$, $r \leftarrow 1$, $s \leftarrow 1$
5. Remove $\pi(r)$-th link from *Adj_l*
6. $a_i \leftarrow$*connected*(t_i), $i = 1, \ldots, k, \quad A \leftarrow \{a_1, \ldots, a_m\}$
7. $\{a_{i_1}, \ldots, a_{i_{k_1}}\} \leftarrow$*dis_elm*$(a_1, \ldots, a_m)$
8. if $num(A(a_{i_j})) = c(a_{i_j} + 1)$, $j = 1, \ldots, k_1$, then *cluster*$\leftarrow$*cluster*$+c$
9. *state*$\leftarrow$*state_check*(*cluster*)
10. if *state*$=M - s$
    $r_s \leftarrow r$, $r{+}{+}$, $s{+}{+}$, GOTO 5
    else
    $r{+}{+}$, GOTO 5
11. If *state*$=0$ then $N(r_1, \ldots, r_M) \leftarrow N(r_1, \ldots, r_M) + 1$
12. Repeat Steps 4-11 N times
13. Calculate $s_{r_1,\ldots,r_M} = \frac{N(r_1,\ldots,r_M)}{N}$, $r_1 = 1, ..., n$, $r_M = 1, ..., n$.

In our algorithm, the function *perm* simulates a random permutation for numbers $1, 2, \ldots, n$. Steps 6-8 are most important steps in this algorithm because we calculate the number of clusters in these steps which is objective for determining the current state of the network. Function *connected*(t_i) uses BFS algorithm to determine which of network terminals are connected to terminal t_i. For a sequence of real numbers $a_1, \ldots, a_l$, the function *dis_elm*$(a_1, \ldots, a_l)$ is used to indicate which of elements of $a_1, \ldots, a_l$ have different values. The function $num(A(a_i))$ counts the number of elements of A that is equal to a_i.

If we consider all permutations of the links numbers, that is $N = n!$, the result of the algorithm is exact value of the signature. However for $n > 10$, $n!$ is very large number and hence it is not usually possible that we consider all permutations. Then, in these cases, it can be taken into account $N < n!$ of permutations. Since survey on the all permutations is the most time-consuming step in the algorithm and is unavoidable, we apply multthreading parallelization in this step.

The computer program is developed in C++ compiler. To run this program we use a intel core i7 computer with a 2.7 GHZ CPU and 16 GB RAM under Mac OS X 10.11. The experimental results are removed due to the restriction on the pages number.

References

[1] Gertsbakh, I. and Y. Shpungin (2011), *Network Reliability and Resilience*. Springer Briefs, Springer.

[2] Samaniego, F.J. (1985), On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability*, **34**, 69-72.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Estimation of the Bivariate Burr Type III Distribution Parameters Based on Censored Samples

Zazarmi Azizi, A. [1] Sayyareh, A. [2]

[1] Department of Statistics, Razi University of Kermanshah, Iran
[2] Faculty of Mathematics, K.N. Toosi University of Technology, Tehran, Iran

Abstract

We consider the problem of estimation of the parameters of the bivariate Burr type III distribution in presence of random censoring. Since the maximum likelihood estimators of the parameters cannot be expressed in a closed form, we suggest to use the EM algorithm to compute the maximum likelihood estimators of the unknown parameters.

Keywords: Bivariate distribution, EM algorithm, Pseudo-likelihood, Random left censoring.

1 Introduction

Many times the life/failure data of interest is bivariate in nature. All studies on twins or on failure data recorded twice on the same system naturally leads to bivariate data.

Burr family of distributions was introduced by Burr (1942). One of the most important distribution of this family is the Burr type III distribution. The probability density function of the Burr type III is given by,

$$f_{BIII}(x;c,k) = kcx^{-c-1}(1+x^{-c})^{-k-1}, \qquad x>0.$$

Here $c>0$ and $k>0$ are the two shape parameters.

Suppose $U_1 \sim BIII(c,k_1)$, $U_2 \sim BIII(c,k_2)$ and $U_3 \sim BIII(c,k_3)$ and they are mutually independent. Let $X_1 = \max(U_1,U_3)$ and $X_2 = \max(U_2,U_3)$. Then (X_1,X_2) has a bivariate Burr

[1] zazamiazizi.afsaneh@stu.razi.ac.ir
[2] asayyareh@kntu.ac.ir

type III distribution with parameters c, k_1, k_2, and k_3 is expressed as $BBIII(c, k_1, k_2, k_3)$. The, joint density function of (X_1, X_2) is given as

$$f(x,y) = \begin{cases} f_{BIII}(x; c, k_1 + k_3) f_{BIII}(y; c, k_2) & if \quad x < y \\ f_{BIII}(x; c, k_1) f_{BIII}(y; c, k_2 + k_3) & if \quad x > y \\ f_{BIII}(x; c, k_1 + k_2 + k_3) & if \quad x = y \end{cases}$$

Suppose the pair (X_1, X_2) is subject to random left censoring by an independent pair of random variables (Y_1, Y_2). We observe $(T_1, \delta_1; T_2, \delta_2)$ where $T_1 = \max(X_1, Y_1), \delta_1 = I(X_1 > Y_1)$ and $T_2 = \max(X_2, Y_2), \delta_2 = I(X_2 > Y_2)$. Therefore, if $X_1 < Y_1$, X_1 is censored.
In order to write down the likelihood, we note that, when $\delta_1 = \delta_2 = 1$, both failure times are observed and the contribution to the likelihood is $f(t_1, t_2)$. When $\delta_1 = 1 - \delta_2 = 1$, first component fails at t_1 and the second component is censored (fails before t_2) and the contribution to the likelihood is $\int_0^{t_2} f(t_1, y)dy$. Similarly, when $1 - \delta_1 = \delta_2 = 1$, first component is censored (fails before t_1) and the second component fails at t_2 and the contribution to the likelihood is $\int_0^{t_1} f(x, t_2)dx$. Finally, when $1 - \delta_1 = 1 - \delta_2 = 1$, both failure times are censored and the contribution to the likelihood is $F(t_1, t_2)$. Hence, the likelihood function, based on $(T_{1i}, \delta_{1i}; T_{2i}, \delta_{2i})$, $i = 1, 2, ..., n$ is given by

$$\begin{aligned} L &= \prod_{i=1}^{n} \big[f(t_{1i}, t_{2i})\big]^{\delta_{1i}\delta_{2i}} \big[\int_0^{t_{2i}} f(t_{1i}, y)dy\big]^{\delta_{1i}(1-\delta_{2i})} \\ &\quad \big[\int_0^{t_{1i}} f(x, t_{2i})dx\big]^{(1-\delta_{1i})\delta_{2i}} \big[F(t_{1i}, t_{2i})\big]^{(1-\delta_{1i})(1-\delta_{2i})}. \end{aligned}$$

Let I_0, I_1, I_2, denote the following sets

$$I_0 = \{i | t_{1i} = t_{2i} = t_i\} \qquad I_1 = \{i | t_{1i} < t_{2i}\} \qquad I_2 = \{i | t_{1i} > t_{2i}\}.$$

Then the likelihood function can be written as

$$L = \prod_{i \in I_0} L(t_i, \delta_{1i}, t_i, \delta_{2i}) \prod_{i \in I_1} L(t_{1i}, \delta_{1i}, t_{2i}, \delta_{2i}) \prod_{i \in I_2} L(t_{1i}, \delta_{1i}, t_{2i}, \delta_{2i}).$$

Let n_0,n_1,n_2, respectively, denote the number of elements in the sets I_0,I_1,I_2 and n_{ij} be the number of pairs for which $(\delta_1, \delta_2) = (i, j), i, j = 0, 1$. Then, $n = \sum_{i=0}^{2} \sum_{j=0}^{2} n_{ij}$ and $n_k = \sum_{i=0}^{2} \sum_{j=0}^{2} n_{ij}^k$, $k = 0, 1, 2$ where n_{ij}^k denotes the number of individuals in I_k with $(\delta_1, \delta_2) = (i, j), i, j = 0, 1, k = 0, 1, 2$.

2 EM algorithm under random censoring

Maximizing the likelihood with respect to k_1 , k_2 , k_3, and c is a non linear optimization problem. We suggest to use of ECM (Expectation Conditional Maximization) algorithm for finding the M.L.E.s of the unknown parameters. It is easy to see that

$$\mu_1 = P(U_1 < U_3 < U_2 | X_1 < X_2) = \frac{k_3}{k_1 + k_3}, \quad \mu_2 = P(U_3 < U_1 < U_2 | X_1 < X_2) = \frac{k_1}{k_1 + k_3}$$
$$\nu_1 = P(U_2 < U_3 < U_1 | X_1 > X_2) = \frac{k_3}{k_2 + k_3}, \quad \nu_2 = P(U_3 < U_2 < U_1 | X_1 > X_2) = \frac{k_2}{k_2 + k_3}.$$

Each of the sets I_0,I_1,I_2 contributes to the log-likelihood function of the pseudo data based on E step. Let

$$\begin{aligned} N_0 &= n_{11}^0 + n_{10}^0 + n_{01}^0 + 2n_{11}^1 + n_{10}^1 + n_{01}^1 + 2n_{11}^2 + n_{10}^2 + n_{01}^2 \\ N_1 &= n_{10}^0 + \mu_2(n_{11}^1 + n_{10}^1) + (n_{11}^2 + n_{10}^2) \\ N_2 &= n_{01}^0 + n_{11}^1 + n_{01}^1 + \nu_2(n_{11}^2 + n_{01}^2) \\ N_3 &= n_{11}^0 + \mu_1(n_{11}^1 + n_{10}^1) + \nu_1(n_{11}^2 + n_{01}^2). \end{aligned} \tag{2.1}$$

Using equations (2.1), the pseudo log-likelihood is given by

$$\begin{aligned}
N_0 \quad & \log c + N_1 \log k_1 + N_2 \log k_2 + N_3 \log k_3 \\
- \quad & k_1\big[\sum_{i\in I_0}\log(1+t_i^{-c}) + \sum_{i\in I_1}\log(1+t_{1i}^{-c}) + \sum_{i\in I_2}(1-\delta_{2i}+\delta_{1i}\delta_{2i})\log(1+t_{1i}^{-c})\big] \\
- \quad & k_2\big[\sum_{i\in I_0}\log(1+t_i^{-c}) + \sum_{i\in I_1}(1-\delta_{1i}+\delta_{1i}\delta_{2i})\log(1+t_{2i}^{-c}) + \sum_{i\in I_2}\log(1+t_{2i}^{-c})\big] \\
- \quad & k_3\big[\sum_{i\in I_0}\log(1+t_i^{-c}) + \sum_{i\in I_1}\log(1+t_{1i}^{-c}) + \sum_{i\in I_2}\log(1+t_{2i}^{-c})\big] \\
- \quad & (c+1)\big[\sum_{i\in I_0}(\delta_{1i}+\delta_{2i}-\delta_{1i}\delta_{2i})\log t_i + \sum_{i\in I_1\cup I_2}(\delta_{1i}\log t_{1i} + \delta_{2i}\log t_{2i})\big] \\
- \quad & \sum_{i\in I_0}(\delta_{1i}+\delta_{2i}-\delta_{1i}\delta_{2i})\log(1+t_i^{-c}) - \sum_{i\in I_1\cup I_2}\big[\delta_{1i}\log(1+t_{1i}^{-c}) + \delta_{2i}\log(1+t_{2i}^{-c})\big] \\
+ \quad & \sum_{i\in I_1}\delta_{1i}(1-\delta_{2i})\log\big[(1+t_{2i}^{-c})^{-k_2} - (1+t_{1i}^{-c})^{-k_2}\big] \\
+ \quad & \sum_{i\in I_2}(1-\delta_{1i})\delta_{2i}\log\big[(1+t_{1i}^{-c})^{-k_1} - (1+t_{2i}^{-c})^{-k_1}\big] \qquad (2.2)
\end{aligned}$$

In order to implement the M-step of the EM algorithm, we need to maximize the pseudo log likelihood equation (2.2) w.r.t k_1, k_2, k_3 and c

$$\begin{aligned}
\frac{\partial \ell}{\partial k_1} &= \frac{N_1}{k_1} + \big[\sum_{i\in I_0}\log(1+t_i^{-c}) + \sum_{i\in I_1}\log(1+t_{1i}^{-c}) + \sum_{i\in I_2}(1-\delta_{2i}+\delta_{1i}\delta_{2i})\log(1+t_{1i}^{-c})\big] \\
&- \sum_{i\in I_2}(1-\delta_{1i})\delta_{2i}\frac{\ln(1+t_{1i}^{-c})(1+t_{1i}^{-c})^{-k_1} - \ln(1+t_{2i}^{-c})(1+t_{2i}^{-c})^{-k_1}}{(1+t_{1i}^{-c})^{-k_1} - (1+t_{2i}^{-c})^{-k_1}} = 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial k_2} &= \frac{N_2}{k_2} + \big[\sum_{i\in I_0}\log(1+t_i^{-c}) + \sum_{i\in I_1}(1-\delta_{1i}+\delta_{1i}\delta_{2i})\log(1+t_{2i}^{-c}) + \sum_{i\in I_2}\log(1+t_{2i}^{-c})\big] \\
&- \sum_{i\in I_1}\delta_{1i}(1-\delta_{2i})\frac{\ln(1+t_{2i}^{-c})(1+t_{2i}^{-c})^{-k_2} - \ln(1+t_{1i}^{-c})(1+t_{1i}^{-c})^{-k_2}}{(1+t_{2i}^{-c})^{-k_2} - (1+t_{1i}^{-c})^{-k_2}} = 0,
\end{aligned}$$

$$\frac{\partial \ell}{\partial k_3} = \frac{N_3}{k_3} + \big[\sum_{i\in I_0}\log(1+t_i^{-c}) + \sum_{i\in I_1}\log(1+t_{1i}^{-c}) + \sum_{i\in I_2}\log(1+t_{2i}^{-c})\big] = 0,$$

and

$$\begin{aligned}
\frac{\partial \ell}{\partial c} &= \frac{N_0}{c} - (k_1+k_2+k_3)\sum_{i\in I_0}\frac{\ln t_i t_i^{-c}}{1+t_i^{-c}} - (k_1+k_3)\sum_{i\in I_1}\frac{\ln t_{1i}t_{1i}^{-c}}{1+t_{1i}^{-c}} \\
&- (k_2+k_3)\sum_{i\in I_2}\frac{\ln t_{2i}t_{2i}^{-c}}{1+t_{2i}^{-c}} + \sum_{i\in I_1}(1-\delta_{2i}+\delta_{1i}\delta_{2i})\frac{\ln t_{1i}t_{1i}^{-c}}{1+t_{1i}^{-c}} \\
&+ \sum_{i\in I_2}(1-\delta_{1i}+\delta_{1i}\delta_{2i})\frac{\ln t_{2i}t_{2i}^{-c}}{1+t_{2i}^{-c}} - \sum_{i\in I_0}(\delta_{1i}+\delta_{2i}-\delta_{1i}\delta_{2i})\log t_i \\
&- \sum_{i\in I_1\cup I_2}[\delta_{1i}\log t_{1i} + \delta_{2i}\log t_{2i}] - \sum_{i\in I_0}(\delta_{1i}+\delta_{2i}-\delta_{1i}\delta_{2i})\frac{\ln t_i t_i^{-c}}{1+t_i^{-c}} \\
&+ \sum_{i\in I_1\cup I_2}[\frac{\delta_{1i}\ln t_{1i}t_{1i}^{-c}}{1+t_{1i}^{-c}} + \frac{\delta_{2i}\ln t_{2i}t_{2i}^{-c}}{1+t_{2i}^{-c}}] \\
&+ \sum_{i\in I_1}\delta_{1i}(1-\delta_{2i})\frac{k_2\ln t_{2i}t_{2i}^{-c}(1+t_{2i}^{-c})^{-k_2-1} - k_2\ln t_{1i}t_{1i}^{-c}(1+t_{1i}^{-c})^{-k_2}}{(1+t_{2i}^{-c})^{-k_2} - (1+t_{1i}^{-c})^{-k_2}} \\
&+ \sum_{i\in I_2}(1-\delta_{1i})\delta_{2i}\frac{k_1\ln t_{1i}t_{1i}^{-c}(1+t_{1i}^{-c})^{-k_1-1} - k_1\ln t_{2i}t_{2i}^{-c}(1+t_{2i}^{-c})^{-k_1}}{(1+t_{1i}^{-c})^{-k_1} - (1+t_{2i}^{-c})^{-k_1}} = 0.
\end{aligned}$$

We observe that there is no explicit solution of any of the M. L. equations as parameters are interrelated. To maximize the pseudo log-likelihood function we use the method of fixed point equation.

3 Discussion

In this paper we have considered the M.L.E.s of the four parameters of bivariate Burr type III distribution when both components of the bivariate variable are subject to random censoring. Since the estimators can not be expressed in a closed form, we suggest to use of expectation-conditional maximization algorithm.

References

[1] Kundu, D. and Dey, A.K., (2009), Estimating the parameters of the Marshall Olkin Bivariate Weibull distribution by EM algorithm, *Computational Statistics and Data Analysis* , **53 (4)**, 956-965.

[2] Kundu, D. and Gupta, R.D., (2009), Bivariate generalized exponential distribution, *Journal of Multivariate Analysis*, **100**, 581-593.

[3] Nandi, S. and Dewan , I., (2010), An EM algorithm for estimating the parameters of bivariate Weibull distribution under random censoring, *Computational Statistics and Data Analysis*, **54**, 1559-1569.

[4] Zazarmi Azizi, A., Sayyareh, A. and Panahi, H., (2013), Inference about the Burr Type III Distribution under Type-II Hybrid Censored Data, *Statistical Research and Training Center*, **10**, 125-149.

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Inference for Lindley distribution under different censoring schemes

Akbar Asgharzadeh [1]

[1] Department of Statistics, University of Mazandaran

Abstract

In the recent years, Lindley distribution has received a considerable attention in the statistical literature. In this talk, pivotal, likelihood and Bayesian inferences are discussed for estimating the unknown parameter of the Lindley distribution based on different censoring schemes. We propose a new method based on a pivotal quantity to estimate the unknown parameter. Maximum likelihood and Bayes estimators are also discussed. Different confidence intervals are considered to estimate the unknown parameter. We also discuss the prediction of future failures based on observed censored data. Finally, Monte Carlo simulations are performed to compare the performances of the different methods, and one data analysis has been presented for illustrative purposes.

Keywords: Censoring schemes, Estimation, Lindley distribution, Monte Carlo simulation

[1] a.asgharzadeh@umz.ac.ir

Second Seminar on
Reliability Theory and its Applications
18-19, May 2016

Mean residual life function of a coherent system with known number of failed components

Khanjari Sadegh, M. [1] Mirjalili, A. [2]

[1,2] Department of Statistics, Faculty of Mathematical Sciences and Statistics, University of Birjand, Birjand, Iran

Abstract

This article considers the mean residual life (MRL) function of a coherent system when the number of failed components of the system is known. We assume that the lifetimes of the system components are independent and identically distributed (iid) random variables and show that when the components of the system have increasing (decreasing) failure rate, that is IFR (DFR), the MRL function of the system is decreasing (increasing) in time. Same property for k-out-of-n systems with independent but not identical (inid) components is also shown when the failed components of the system are known, extending some results in literatures.

Keywords: Coherent system, mean residual lifetime, signature, IFR, DFR

1 Introduction

The mean residual life and the failure rate functions are very important in Reliability and Survival analysis. It is well known that both of them uniquely determine the distribution function(see e.g. Barlow and Proschan, 1975). Hence in recent years, the MRL functions of technical systems, such as k-out-of-n systems, have been widely studied by the many authors and under various assumptions. For example Khanjari (2008-a) studied the MRL function of a parallel system with inid components. Asadi and Goliforushani (2008) obtained some properties of the MRL function of a coherent system with iid components. Their results have been extended to the systems with inid or exchangeable components by Khanjari (2011). Let $T_1, \ldots, T_n$ denote the lifetimes of n components which are connected in a coherent system and let $T = \phi(T_1, \ldots, T_n)$ represent the lifetime of the system. For a details on the coherent structures see e.g. Barlow and Proschan

[1]mkhanjari@birjand.ac.ir
[2]

(1975). We assume that T_i's are iid random variables and $F(t) = 1 - \bar{F}(t)$ is their common distribution function. Suppose $N(t)$ is the number of failed components of the system up to time t. We note that $N(t)$ is distributed as Binomial$(n, F(t))$. We define $M_r(t)$, the MRL function of the system as follow:

$$M_r(t) = E(T - t|T > t, N(t) = r) \tag{1.1}$$

Let $T_{i:n}$ represent the ith ordered lifetime of components. It is well known that the lifetime of a k-out-of-n:F system is $T_{k:n}$. Khanjari (2008-b) defined

$$H^r_{n,k}(t) = E(T_{k:n} - t|N(t) = r), \;\; 0 \le r < k \le n, \; t > 0 \tag{1.2}$$

and showed that if $H^k_n(t) = E(T_{k:n} - t|T_{1:n} > t)$ then

$$H^r_{n,k}(t) = E(T_{k-r:n-r} - t|T_{1:n-r} > t) = H^{k-r}_{n-r}(t) \tag{1.3}$$

In fact

$$P(T_{k:n} > t + x|N(t) = r) = P(T_{k-r:n-r} > t + x|T_{1:n-r} > t)$$

He also characterized $\bar{F}(t)$ by $H^r_{n,k}(t)$. In the following section we extend his result to a coherent system. Also in inid case, the MRL function of a k-out-of-n:F system is considered when the both $N(t)$ and the failed components of the system are known.

2 Main results

Let $T = \phi(T_1, \ldots, T_n)$, $T_{i:n}$, $M_r(t)$, $N(t)$, $H^k_n(t)$ and $H^r_{n,k}(t)$ are defined as in previous section. Samaniego (1985) obtained the following interesting and important result

$$P(T > t) = \sum_{i=1}^{n} s_i P(T_{i:n} > t) \tag{2.4}$$

where $s_i = P(T = T_{i:n})$ and called the probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ as the signature of the system. In the following lemma it is shown that the MRL function $M_r(t)$ can be written as a convex combination of MRL's $H^{k-r}_{n-r}(t)$, $k = r + 1, \ldots, n$.

For $0 \le r < k \le n$ and $t > 0$ we have

$$M_r(t) = \sum_{k=r+1}^{n} w_k H^{k-r}_{n-r}(t) \tag{2.5}$$

where $w_k = s_k / \sum_{r+1}^{n} s_j$, $k = r + 1, \ldots, n$. **Proof.** We have $P(T > t + x|T > t, N(t) = r) = \frac{P(T>t+x|N(t)=r)}{P(T>t|N(t)=r)}$ It is known that for $1 \le i, j \le n$ two events $T = T_{j:n}$ and $T_{i:n}$ are independent(see e.g. Kochar et al. 1999). Note that $N(t) = r$ is equivalent to $T_{r:n} < t < T_{r+1:n}$. Therefore two events $N(t) = r$ and $T \ge T_{r+1:n}$ are also independent. Now in view of Lemma 2.2 in Samaniego et al. (2009) and from Equations (1.2) and (1.3) we have

$$P(T > t + x|T > t, N(t) = r) = \frac{\sum_{k=r+1}^{n} s_k P(T_{k:n} > t + x|N(t) = r)}{\sum_{j=r+1}^{n} s_j}$$

which is equal to $\sum_{k=r+1}^{n} w_k P(T_{k-r:n-r} > t + x|T_{1:n-r} > t)$. Now by integrating both sides of this equation with respect to x the proof of the lemma is immediate.

Remark 2.1. Samaniego et al.(2009) showed that $P(T = T_{k:n}|T > t, N(t) = r) = s_k / \sum_{r+1}^{n} s_j = w_k$, $k = r + 1, \ldots, n$ and called the probability vector $\mathbf{w}_{(n-r)} = (w_{r+1}, \ldots, w_n)$ as the dynamic signature of the system. By using other argument they showed that $P(T > t + x|T > t, N(t) = r) = \sum_{j=1}^{n-r} w_{j+r} \bar{G}_{j:n-r|t}(x)$ where $\bar{G}_{j:n-r|t}(x)$ is the reliability function of the jth order statistic from a random sample of size $n - r$ from $\bar{G}(x|t) = \bar{F}(x + t)/\bar{F}(t)$. As shown in Khanjari (2008-b), we note that $\bar{G}_{j:n-r|t}(x) = P(T_{j:n-r} > t + x|T_{1:n-r} > t) = P(T_{j+r:n} > t + x|N(t) = r)$.

Remark 2.2. Asadi and Goliiforushani (2008) proved that $H_n^k(t) = E(T_{k:n} - t|T_{1:n} > t)$ is a decreasing(increasing) function of t, if F is IFR(DFR). (Recall that F is said to be IFR(DFR) if for all $x > 0$, $\bar{F}(t+x)/\bar{F}(t)$ is decreasing(increasing) in t.) In inid case Khanjari (2011) showed that when F_i's are IFR(DFR), $H_n^k(t)$ is decreasing(increasing) in t. He also showed that$(T_{k:n} - t|T_{1:n} > t) \leq_{st} (T_{k:n-1} - t|T_{1:n-1} > t)$ and therefore $H_n^k(t) \geq H_{n-1}^k(t)$. Now in iid case and from Lemma 2.1, $M_r(t)$ is decreasing(increasing) in t if F is IFR(DFR).

Remark 2.3. It seems that the probability $p(x,t,r) = P(T > t+x|T > t, N(t) = r)$ is decreasing in $0 \leq r < n$. But this is not true in general. For example suppose $n = 3$ and $\mathbf{s} = (1/3, 1/3, 1/3)$ be the signature of an arbitrary mixed system(see e.g. Samaniego et al.(2009) for a definition of mixed systems). Also suppose $\bar{F}(t+x)/\bar{F}(t) = 0.1$. It is easy to see that $p(x,t,0) = 299/3000 < p(x,t,1) = p(x,t,2) = 0.1$ and if $\mathbf{s} = (1/6, 2/6, 3/6)$ then $p(x,t,0) = 5/8 > p(x,t,1) = 11/20 > p(x,t,2) = 1/2$. Therefore $M_r(t)$ is not monotone in r.

Now consider a k-out-of-n:F system with inid components and suppose $N(t) = r$, $0 \leq r < k \leq n$. We define the set S_r with cardinality of r, as the set of indices of the failed components at or before time t. S_r is a subset of $\{1, \ldots, n\}$ and assume that $S_0 = \emptyset$.

For $t > 0$ and $0 \leq r < k \leq n$ we have

$$E(T_{k:n} - t|N(t) = r, S_r) = H_{S_r'}^{k-r}(t) = E(T_{(k-r:n-r)} - t|T_{(1:n-r)} > t)$$

where $T_{(j:n-r)}$ is the jth ordered lifetime of the random sample $\{T_i|i \in S_r' = \{1, \ldots, n\} - S_r\}$

Proof. It is easy to show that

$$P(T_{k:n} > t + x|N(t) = r, S_r) = P(T_{(k-r:n-r)} > t + x|T_{(1:n-r)} > t).$$

Now the proof of the lemma is immediate.

Remark 2.4. In view of Remak 2.2 we note that in inid case, when F_i's are IFR(DFR), the MRL function of a k-out-of-n:F system, given in Lemma 2.2, is decreasing(increasing) in t.

REFERENCES

[1] Barlow, R.E. and Proschan, F. (1975), *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston.

[2] Asadi, M. and Goliforushani, S. (2008), On the mean residual life function of coherent systems. *IEEE Trans. on Rel.*, **57(4)**, 574-580.

[3] Khanjari, S.M. (2008-a), Mean past and mean residual life functions of a parallel system with nonidentical components. *Commu. in Stat.; Theory & Methods*, **37**, 1134-1145.

[4] Khanjari, S.M. (2008-b), A note on the regression of order statistics. *Commu. in Stat.; Theory & Methods*, **37**, 475-480.

[5] Khanjari, S.M. (2011), A note on the mean residual life function of a coherent system with exchangeable or nonidentical components. *Journal of Statistical Planning and Inference*, **141**, 3267-3275.

[6] Samaniego, F.J. (1985), on the closure of the IFR class under formation of coherent systems. *IEEE Trans. on Rel.*, **34**, 69-72.